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UNIVERSITY OF NEW MEXICO  
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# ENGINEERING EXPERIMENT STATION

THE FIELDS OF ELECTRIC DIPOLES IN SEA WATER --  
THE EARTH-AIR-IONOSPHERE PROBLEM

EE-88

by

Wallace L. Anderson

May 1963

This work was performed  
under Contract Nonr 2798(01)

Engineering Experiment Station  
University of New Mexico  
Albuquerque, New Mexico

THE FIELDS OF ELECTRIC DIPOLES IN SEA WATER --  
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Wallace L. Anderson\*

This is a corrected version of Technical Report EE-44,  
published in February 1961.

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## PREFACE

The values of the field strength given in the section on numerical comparison (5.1 and 5.2) of the original publication of this report were found to be in error. Dr. Paul S. Swan\* first suggested the possibility of an error and Dr. James R. Wait\*\* showed that such an error existed. Subsequently two independent evaluations of these numerical values were made. These were in close agreement. The results of these evaluations are published in this "corrected edition" of Technical Report EE-44. Using these revised field strengths, comparison of the relative efficiencies for the submerged horizontal and vertical dipoles shows that the horizontal submerged dipole is more effective than the vertical submerged dipole in exciting a vertically-polarized field in the air.

For a receiving site located 1000 km from a transmitting antenna with 100 kw of input power at 1 cps, a signal-to-noise ratio of approximately -13 db is predicted by the mode solution. At closer distances the branch-line solution indicates that the signal is considerably stronger, thus suggesting the possibility of practical communications at distances less than 1000 km.

This "corrected edition" of the report supersedes the original report (Technical Report EE-44), dated February 1961, which should be destroyed and replaced by this "corrected edition."

---

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\*\* National Bureau of Standards, Boulder, Colorado.

## 1.0 INTRODUCTION

The radio wave propagation problem as it exists in connection with communications involving submerged submarines is necessarily one which concerns low frequencies. This conclusion is the immediate result of considering the attenuation of an electromagnetic wave as it propagates through a conducting medium, and noting that such attenuation is proportional to a factor  $\exp(-c\sqrt{f} x)$  where  $c$  is a constant for the medium,  $f$  is frequency and  $x$  is distance. When  $x$  is as much as a few meters and the medium concerned is sea water, this factor automatically excludes, for practical purposes, any frequencies above the VLF and ELF categories. It is customary to define the ELF range of frequencies as extending roughly from 1 to 3000 cps<sup>1</sup> and VLF from 3,000 to 30,000 cps.<sup>2</sup>

In the initial part of the following investigation, expressions will be obtained for the Hertz potential in three regions of arbitrary conductive and dielectric properties. The regions are separated from each other by two infinite parallel planes. In directions normal to these planes, the regions above and below the intermediate layer are considered to be of

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<sup>1</sup>J. R. Wait and Nancy F. Carter, Field Strength Calculations for ELF Radio Waves, NBS Tech. Note No. 52, Boulder Laboratories, March 1960.

<sup>2</sup>J. R. Wait, A Survey and Bibliography of Recent Research in the Propagation of VLF Radio Waves, NBS Tech. Note No. 58, Boulder Laboratories, May 1960.

semi-infinite extent. The regions are individually homogeneous. An electric dipole is located in one of the semi-infinite media, which will be called medium 1. The intermediate layer is called medium 2, and the other medium 3.

Following this initial formulation, specialization will be made to the case where medium 1 is sea water, medium 2 is air, and medium 3 is ionosphere. Under these conditions the ionosphere will form the upper boundary of a parallel plane waveguide having for its lower boundary the surface of the sea.

The theory that is developed is valid, up to a certain point, for antennas buried in earth. The point at which a departure is made to the specific case of sea water is in the evaluation of the branch line integrals, where certain approximations are made based upon the magnitudes of the parameters. It would not be difficult to perform the different evaluations necessary for antennas in earth, but this has not been undertaken in this paper.

The choice of a parallel plane geometry is dictated by reasons of simplicity. There is no intrinsic reason why equations, however involved, cannot be worked out for a spherical model. There are, however, two points which argue against developing the more formidable case first. One is the practical difficulty of extracting numerical information from the results, which would involve the poorly tabulated spherical Hankel functions of complex order and argument. The other is the fact that the parallel plane development will give a reasonably good estimate out to distances from the antenna on the order of the earth's

radius, and this is an ample range from which to decide whether a spherical model approach, required for the greater ranges, is practically justified.<sup>1</sup>

In the following, both vertical and horizontal submerged electric dipoles are considered. In each case a "mode" solution is obtained, corresponding to TEM-type propagation in the parallel plane waveguide for which the boundaries are water and ionosphere. Restriction of the frequency range to 1-1000 cps avoids the necessity of discussing the higher order modes, which are then so heavily damped as to be of no practical significance.

Extension of the results to higher frequencies is nonetheless readily carried out, provided only that the complex characteristic numbers (occurring, in our development, as poles in the complex plane) can be found for the modes in which appreciable transmission can occur.

Investigators at the Boulder Laboratories of the National Bureau of Standards have apparently been successful in finding the important characteristic numbers in connection with standard VLF and ELF mode propagation theory.<sup>3</sup> Since the modes are the same no matter how they are excited, the pole locations necessary for consideration of higher frequencies are therefore available and immediately applicable to the results of this paper.

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<sup>3</sup>Howe, H. H. and Wait, J. R., "Mode Calculations for VLF Ionospheric Propagation", Proc. Symposium on Propagation of VLF Radio Waves, V. 3, paper no. 36, Boulder, Colorado, 1957.

In addition to fields from the mode solutions, there are field contributions due to the integration around two branch cuts in the complex plane. The relative contributions from these branch cuts can be shown to diminish rapidly with increasing distance from the real axis of this complex plane, this distance being proportional to the square root of conductivity times frequency. In the pages that follow, we find that the integral for one of these cuts--that connected with the propagation constant for a plane wave in sea water--is practically negligible at large distances from the antenna. Physically this means that propagation directly through the water from transmitting to receiving antenna can be disregarded.

The contribution from the branch cut associated with the propagation constant for a plane wave in the ionosphere is not important at distances out to around 1000 kilometers for the lower part of the ELF range. To a first approximation this contribution behaves asymptotically as  $e^{-a\rho}/\rho^2$ , where  $\rho$  is distance in meters. The factor  $\sqrt{F}$  is contained in the quantity  $a$ , which is the imaginary part of the propagation constant  $k_z$ , of the ionosphere. Physically, this solution corresponds to propagation along the air-ionosphere boundary, but in the ionosphere.

It is possible to draw some comparisons between the results contained herein, and those which have been obtained by other investigators for the situation when the antennas are located in the atmosphere. In the latter case the chief interest has been in VLF propagation, for which the branch line integrals are quite rapidly attenuated with distance from the source,

and therefore can be neglected in comparison with the mode solutions. In some cases, however, the VLF mode solutions have been extended to ELF without, apparently, a re-evaluation of the branch line contributions. The writer has given considerable attention to this matter, but the results are necessarily intended to serve only as estimates when the product of distance times square root of frequency is sufficiently great. It is believed that these estimates--in spite of the uncertainty associated with them--have served the very useful purpose of showing that the branch line solution--at least in the case of the horizontal submerged dipole--is behaving in such a way as to merit close attention in the low frequency part of the ELF range, at distances less than 1000 km.

This possible importance of the branch line solutions justifies a more thorough investigation of their magnitudes. A different approach is then required, and it appears that a possible line of attack is to place the branch cut from  $k_3$  at an angle of  $-45^\circ$  (in the direction of  $k_1$  -- see Figure 2.2). Then tabulated values of the Bessel functions of argument  $(e^{-i\pi/4}x)$  may be used to effect a numerical integration. This work would perhaps be best performed by means of an automatic computer. It will be found later (Section 6.0) that there may be some hope of useful communications between submerged antennas at 1000 km. This preliminary conclusion is based, however, on some rather optimistic data which has been published for natural noise in this frequency range. The more exact evaluation of the branch line integrals should therefore perhaps be confined, at

least initially, to distances of 1000 km or less, which might constitute a less borderline situation.

It has been found that for a submerged short-circuited horizontal coaxial antenna 540 meters long and a transmitter power of 100 kilowatts, the signal power at 1000 km distance can be expected to be about 13 db below noise power for 1 cps bandwidth and a center frequency of 1 cps excluding the exponential depth factor. This bandwidth is quite large at this frequency; hence improvement is possible, depending on how much bandwidth is required for the information being communicated. At a distance of 100 km, the signal power would be roughly 40 db greater due to the mode solution; an additional contribution might come from the branch line integral as well.

A few words should be said about the role of the ionosphere in VLF and ELF radio wave propagation. It is apparently permissible to ignore the presence of the ionosphere at distances less than about 30-50 kilometers, in considering the fields near the ground.<sup>1</sup> Extension of the results of this paper to such distances, by numerical integration of the branch line integral, would therefore enable comparison of these results with those obtained by R. K. Moore<sup>4</sup> and by Baños and Wesley.<sup>5</sup> Their analyses were concerned with submerged antennas when the atmosphere is assumed homogeneous.

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<sup>4</sup>R. K. Moore, Doctoral Dissertation, The Theory of Radio Communications Between Submerged Submarines, Cornell Univ. 1951.

<sup>5</sup>A. Baños, Jr., and J. P. Wesley, The Horizontal Dipole in a Conducting Half-Space, University of California Marine Physical Laboratory, Reports No. 53-33, 1953, and No. 54-33, 1954.

## 2.0 THE SUBMERGED VERTICAL ELECTRIC DIPOLE

### 2.1 Introduction of Equations

The development is in terms of the Hertz vector,  $\vec{\Pi}$ , from which the electric and magnetic fields are derivable by the relations:

$$\vec{B} = \mu \bar{\epsilon} \nabla \times \frac{\partial \vec{\Pi}}{\partial t} \quad (2.1-1)$$

$$\vec{E} = \nabla \nabla \cdot \vec{\Pi} - \mu \bar{\epsilon} \frac{\partial^2 \vec{\Pi}}{\partial t^2} \quad (2.1-2)$$

where  $\bar{\epsilon} = (\epsilon + \frac{\sigma}{i\omega})$  is the so-called complex dielectric constant, and  $\vec{\Pi}$  satisfies the vector equation

$$\nabla \times \nabla \times \vec{\Pi} - \nabla \nabla \cdot \vec{\Pi} + \mu \bar{\epsilon} \frac{\partial^2 \vec{\Pi}}{\partial t^2} = 0 \quad (2.1-3)$$

The constant  $\epsilon$  is given by  $k\epsilon_0$ , where  $k$  is the ordinary dielectric constant, and  $\epsilon_0$  is given by  $10^7/4\pi c^2$ , with  $c$  = velocity of light in free space. Conductivity is represented by  $\sigma$ , while  $\omega$  is  $2\pi f$ .

These equations are applicable in regions which, individually are homogeneous and isotropic, and in which no free charges are present. In the remainder of our discussion it will be assumed that  $\vec{\Pi}(x,y,z,t) = \vec{\Pi}(x,y,z) e^{i\omega t}$ . The time factor  $e^{i\omega t}$  will be suppressed, and the Hertz vector will be understood to be a function of position only. With this modification, equation (2.1-3) can be written:

$$\nabla \times \nabla \times \vec{\Pi} - \nabla \nabla \cdot \vec{\Pi} - k^2 \vec{\Pi} = 0 \quad (2.1-4)$$

where  $k^2 = \mu \bar{\epsilon} \omega^2 = \mu \epsilon \omega^2 - i \omega \mu \sigma$ . If the operator  $\nabla^2$  is understood as acting on rectangular components of  $\vec{\Pi}$ , we use a common vector identity to simplify further:

$$\nabla^2 \vec{\Pi} + k^2 \vec{\Pi} = 0 \quad (2.1-5)$$

Thus the rectangular components of  $\vec{\Pi}$  individually satisfy the homogeneous equation, in source-free regions. To take account of the source, it is necessary only to add solutions of (2.1-5) to a solution of the inhomogeneous wave equation, in such a way that the vector is singular at the source in the prescribed manner. In the present problem this is accomplished by designating (see Figure 2.1) for medium 1:

$$\vec{\Pi}_1 = \vec{\Pi}_{(pri)} + \vec{\Pi}_1 (sec) \quad (2.1-6)$$

where  $\vec{\Pi}_{(pri)}$  satisfies the inhomogeneous wave equation, and  $\vec{\Pi}_{(sec)}$  the homogeneous wave equation.  $\vec{\Pi}_2$  and  $\vec{\Pi}_3$  have only to satisfy the homogeneous equations as written for their respective media:

$$\nabla^2 \vec{\Pi}_n + k_n^2 \vec{\Pi}_n = 0 \quad n = 2, 3 \quad (2.1-7)$$

In order that  $\vec{\Pi}_{(pri)}$  represent properly the field of the electric dipole source oriented along the z-axis, it is written specifically as :

$$\vec{\Pi}_{(pri)} = \frac{I dl}{4\pi i \omega \bar{\epsilon}} \frac{e^{-ik_1 R_1}}{R_1} \vec{k} \quad (2.1-8)$$

with  $\vec{k}$  the unit vector in the z-direction. In cylindrical coordinates obviously  $R_1 = \sqrt{\rho^2 + (z+z_0)^2}$ . The problem has cylindrical symmetry, and is subject to the boundary conditions:

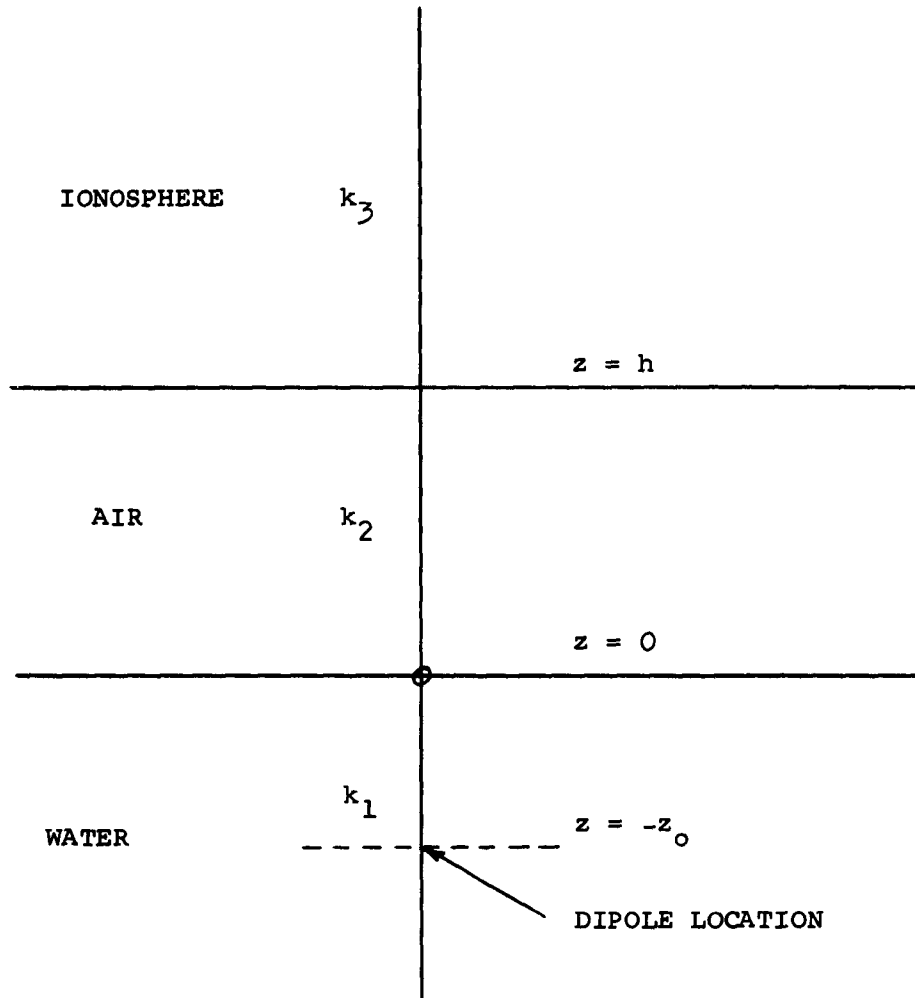


Figure 2.1

$$\begin{aligned}
 \underline{z} = 0 \quad \vec{k} \times (\vec{E}_2 - \vec{E}_1) &= 0 \\
 \vec{k} \times (\vec{H}_2 - \vec{H}_1) &= 0 \\
 \underline{z} = h \quad \vec{k} \times (\vec{E}_3 - \vec{E}_2) &= 0 \\
 \vec{k} \times (\vec{H}_3 - \vec{H}_2) &= 0
 \end{aligned} \tag{2.1-9}$$

where the numerical subscripts 1,2,3 denote fields in media 1,2,3, respectively. An additional boundary condition is the so-called "radiation condition":

$$\lim_{R \rightarrow \infty} R \left( \frac{\partial \psi}{\partial R} + ik\psi \right) = 0 \tag{2.1-10}$$

where R is radial distance and  $\psi$  is any of the field components.

We now assume that the fields everywhere can be represented in terms of the z-axis oriented Hertzian potential; if the final solution satisfies the boundary conditions and behaves properly as  $R_1 \rightarrow 0$ , the assumption will have been justified. Consequently, we henceforth drop the vector designation, and write simply " $\Pi$ ", with the orientation understood. In cylindrical coordinates we have:

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \Pi_n}{\partial \rho} \right) + \frac{\partial^2 \Pi_n}{\partial z^2} + \frac{1}{\rho^2} \frac{\partial^2 \Pi_n}{\partial \phi^2} + k_n^2 \Pi_n = 0 \tag{2.1-11}$$

One expects that solutions may be represented as superpositions (discrete or continuous) of cylindrical wave functions:

$$\psi_n(\rho, \phi, z) = e^{\pm im\phi} \left\{ \begin{matrix} J_m \\ N_m \end{matrix} \right\}(\lambda \rho) e^{\pm \sqrt{k_n^2 - \lambda^2} z} \tag{2.1-12}$$

where  $\lambda$  and  $m$  are separation parameters.

Since the solutions are required to possess azimuthal symmetry, and must be finite at  $\rho = 0$ , we choose the Bessel functions of the first kind and  $m=0$ . Thus the desired functions are:

$$J_0(\lambda\rho) e^{\pm i\sqrt{k_n^2 - \lambda^2} z} \quad (2.1-13)$$

It will be convenient to use  $\beta_n$ , defined by  $\beta_n^2 = k_n^2 - \lambda^2$ , as a new separation parameter.

In the three regions we may now anticipate the representations:

$$\Pi_1 = M \left\{ \frac{e^{-ik_1 R_1}}{R_1} + \int b_1(\lambda) J_0(\lambda\rho) e^{\pm i\beta_1 z} d\lambda \right\} \quad (2.1-14)$$

$$\Pi_2 = M \int \left[ a_2(\lambda) e^{-i\beta_2 z} + b_2(\lambda) e^{i\beta_2 z} \right] J_0(\lambda\rho) d\lambda \quad (2.1-15)$$

$$\Pi_3 = M \int a_3(\lambda) e^{-i\beta_3 z} J_0(\lambda\rho) d\lambda \quad (2.1-16)$$

where the spectral densities  $b_1$ ,  $b_2$ ,  $a_1$ ,  $a_2$  are to be determined from the boundary conditions, and the limits of integration will be specified later.  $M$  is simply an amplitude factor depending on the "strength" of the antenna, and is described by

$$M = \frac{I dl}{4\pi i\omega\epsilon} \quad (2.1-17)$$

where  $I dl$  represents the dipole moment of the antenna.

In the integrals of (2.1-14) and (2.1-16) a boundary condition has already been applied; namely, that for  $z < -z_0$  and  $z > h$ , only "outgoing" waves are physically allowable. We note in this connection that the sign of the imaginary part of  $\beta$  is to be taken as negative, in order that the integrals converge.

We have already defined  $k^2 = \omega^2 \mu \epsilon - i \omega \mu \sigma$ , in which  $\omega^2 \mu \epsilon$  and  $\omega \mu \sigma$  are positive real numbers. In computing  $k$  we will take the square root which has a positive real (therefore negative imaginary) part. It can then be shown that the primary excitation can be represented as follows:

$$\frac{e^{-ik_1 R_1}}{R_1} = \int_0^\infty \frac{\lambda}{i\beta_1} J_0(\lambda \rho) e^{-i\beta_1 |z+z_0|} d\lambda \quad (2.1-18)$$

It may be noted from figure 2.1 that in the exponent under the integral,  $z_0$  is a positive number. From (2.1-18) we will surmise that the limits of integration in (2.1-14), (2.1-15) and (2.1-16) are also to be taken from zero to infinity. For region 1 we have, finally:

$$\Pi_1 = M \int_0^\infty \left[ \frac{\lambda}{i\beta_1} e^{-i\beta_1 |z+z_0|} + b_1(\lambda) e^{i\beta_1 z} \right] J_0(\lambda \rho) d\lambda \quad (2.1-19)$$

We now apply the boundary conditions. The tangential components of  $\vec{E}$  and  $\vec{H}$  are required to be continuous on the planes  $z = 0$  and  $z = h$ .  $\vec{E}$  and  $\vec{H}$  are derived by the equations

$$\vec{E}_n = \nabla \nabla \cdot \vec{\Pi}_n + k_n^2 \vec{\Pi}_n \quad (2.1-20)$$

$$\vec{H}_n = \frac{k_n^2}{i\omega \mu_n} \nabla \times \vec{\Pi}_n \quad (2.1-21)$$

and are found to be

$$E_{n\rho} = \frac{\partial^2 \Pi_n}{\partial \rho \partial z}; E_{n\phi} = 0; E_{nz} = \left( \frac{\partial^2}{\partial z^2} + k_n^2 \right) \Pi_n$$

$$H_{n\rho} = 0; H_{n\phi} = \frac{k_n^2}{i\omega \mu_n} \frac{\partial \Pi_n}{\partial \rho}; H_{nz} = 0 \quad (2.1-22)$$

An integration of  $E_{n\rho}$  and  $H_{n\theta}$  can be performed from  $\rho$  to infinity. Since the fields and their derivatives are required to vanish at infinity, the boundary conditions are then specified in a somewhat simpler manner by

$$\begin{aligned} \frac{\partial \Pi_n}{\partial z} &= \frac{\partial \Pi_{n+1}}{\partial z} & n=1, z=0 \\ & & n=2, z=h \end{aligned} \quad (2.1-23)$$

$$k_n^2 \Pi_n = k_{n+1}^2 \Pi_{n+1} \quad \begin{aligned} n=1, z=0 \\ n=2, z=h \end{aligned}$$

in which we have assumed  $\mu_n = \mu_0$ , the magnetic permeability of free space, for all three media.

The conditions (2.1-23) are applied to equations (2.1-19), (2.1-15) and (2.1-16). The algebra is somewhat simplified if the coefficients  $b_1$ ,  $a_2$ ,  $b_2$  and  $a_3$  are assumed to contain a factor  $\frac{\lambda}{i\beta_1} e^{-i\beta_1 z_0}$ , thus:

$$\begin{aligned} b_1(\lambda) &= B_1(\lambda) \frac{\lambda}{i\beta_1} e^{-i\beta_1 z_0} \\ a_2(\lambda) &= A_2(\lambda) \frac{\lambda}{i\beta_1} e^{-i\beta_1 z_0} \\ b_2(\lambda) &= B_2(\lambda) \frac{\lambda}{i\beta_1} e^{-i\beta_1 z_0} \\ a_3(\lambda) &= A_3(\lambda) \frac{\lambda}{i\beta_1} e^{-i\beta_1 z_0} \end{aligned} \quad (2.1-24)$$

Four equations are obtained in the four unknowns:

$$\begin{aligned} \beta_1(B_1 - 1) &= \beta_2(B_2 - A_2) \\ \beta_2(B_2 e^{i\beta_2 h} - A_2 e^{-i\beta_2 h}) &= -\beta_3 A_3 e^{-i\beta_3 h} \\ k_1^2(B_1 + 1) &= k_2^2(B_2 + A_2) \end{aligned} \quad (2.1-25)$$

$$k_2^2 (B_2 e^{i\beta_2 h} + A_2 e^{-i\beta_2 h}) = k_3^2 A_3 e^{-i\beta_3 h} \quad (2.1-25)$$

If we define quantities  $K_n(\lambda)$  such that

$$K_n(\lambda) = \frac{i\beta_n}{P_n} \quad (2.1-26)$$

where  $P_n = \sigma_n + i\omega\epsilon_n$ , we can rewrite these equations as

$$\begin{aligned} P_1 K_1 (B_1 - 1) &= P_2 K_2 (B_2 - A_2) \\ P_2 K_2 (B_2 e^{i\beta_2 h} - A_2 e^{-i\beta_2 h}) &= -P_3 K_3 A_3 e^{-i\beta_3 h} \\ P_1 (B_1 + 1) &= P_2 (B_2 + A_2) \\ P_2 (B_2 e^{i\beta_2 h} + A_2 e^{-i\beta_2 h}) &= P_3 A_3 e^{-i\beta_3 h} \end{aligned} \quad (2.1-27)$$

It is then readily determined that

$$B_1 = \frac{K_2(K_1 - K_3) + i(K_1 K_3 - K_2^2) \tan \beta_2 h}{K_2(K_1 + K_3) + i(K_1 K_3 + K_2^2) \tan \beta_2 h} \quad (2.1-28)$$

Then  $K_n(\lambda)$  can be considered as intrinsic wave impedances for the media concerned. It should be pointed out that rewriting  $B_1$  in the form

$$B_1 = \frac{K_1(K_2 + K_3 \tanh i\beta_2 h) - K_2(K_3 + K_2 \tanh i\beta_2 h)}{K_1(K_2 + K_3 \tanh i\beta_2 h) + K_2(K_3 + K_2 \tanh i\beta_2 h)} \quad (2.1-29)$$

leads quickly to the representation

$$B_1 = \frac{K_1 - Z_2}{K_1 + Z_2} \quad (2.1-30)$$

$$\text{where } Z_2 = K_2 \frac{K_3 + K_2 \tanh i\beta_2 h}{K_2 + K_3 \tanh i\beta_2 h} \quad (2.1-31)$$

$Z_2$  is the wave impedance looking into medium 2 at the boundary between 1 and 2.

The other coefficients are found to be

$$\begin{aligned} A_2(\lambda) &= \frac{\beta_1}{\beta_2} \frac{K_2 + Z_2}{K_1 + Z_2} \\ B_2(\lambda) &= \frac{\beta_1}{\beta_2} \frac{K_2 - Z_2}{K_1 + Z_2} \\ A_3(\lambda) &= 2K_3 \frac{\beta_1}{\beta_3} \frac{(K_2 + Z_2) e^{-i\beta_2 h} e^{i\beta_3 h}}{(K_1 + Z_2)(K_2 + K_3)} \end{aligned} \quad (2.1-32)$$

The integrals for the Hertz vector in each region now become:

$$\Pi_1 = M \int_0^\infty e^{-i\beta_1 |z+z_0|} + \frac{K_1 - Z_2}{K_1 + Z_2} e^{i\beta_1 (z-z_0)} \frac{J_0(\lambda \rho) \lambda d\lambda}{i\beta_1} \quad (2.1-33)$$

$$\Pi_2 = 2M \int_0^\infty \frac{1}{i\beta_2} \frac{K_2 \cosh(i\beta_2 z) - Z_2 \sinh(i\beta_2 z)}{K_1 + Z_2} e^{-i\beta_1 z_0} J_0(\lambda \rho) \lambda d\lambda \quad (2.1-34)$$

$$\Pi_3 = \frac{2M}{\sigma_3 + i\omega\epsilon_3} \int_0^\infty \frac{(K_2 + Z_2)}{(K_1 + Z_2)(K_2 + K_3)} e^{-i\beta_1 z_0} e^{-i\beta_2 h} e^{-i\beta_3 (z-h)} J_0(\lambda \rho) \lambda d\lambda \quad (2.1-35)$$

## 2.2 The Hertz Potential in Region 1:

We will first consider region 1:

$$\Pi_1 = M \int_0^\infty G(\beta_1, \beta_2, \beta_3) J_0(\lambda \rho) \lambda d\lambda \quad (2.2-1)$$

where  $G(\beta_1, \beta_2, \beta_3)$  is  $1/i\beta_1$  times the bracketed part of the integrand in equation (2.1-33). We introduce the identity:

$$J_0(\lambda \rho) = 1/2 [H_0^1(\lambda \rho) + H_0^2(\lambda \rho)] \quad (2.2-2)$$

and designate

$$\Pi_1 = (I_1 + I_2) \quad (2.2-3)$$

where

$$I_1 = \frac{M}{2} \int_0^{\infty} H_0^1(\lambda \rho) G(\beta_1, \beta_2, \beta_3) \lambda d\lambda \quad (2.2-4a)$$

$$I_2 = \frac{M}{2} \int_0^{\infty} H_0^2(\lambda \rho) G(\beta_1, \beta_2, \beta_3) \lambda d\lambda \quad (2.2-4b)$$

The functions in this and the other integrands occurring in our problem are analytic except at isolated singularities (poles and branch points), and can be continued into the complex plane when proper account is taken of such singularities.

We therefore transform our integrals along the real axis of  $\lambda$  into contour integrals in the right half of the complex  $\lambda$ -plane. In order that this be feasible, it is necessary that the integrands behave properly when the modulus of  $\lambda$  becomes large. We note that as  $|\lambda| \rightarrow \infty$ ,

$$\beta_n \sim i\lambda.$$

Also,  $K_n(\lambda) \sim A_n \lambda$ ,  $n=1,2,3$ , where the  $A_n$  are constants. The main question now concerns the hyperbolic tangent, which has poles where

$$\cosh(i\beta_2 h) = 0.$$

These occur for

$$i\beta_2 h = \pm(n + \frac{1}{2}) i\pi, \quad n = 0, 1, 2, 3, \dots,$$

which requires

$$\lambda = \pm \sqrt{k_2^2 - \frac{\pi^2}{h^2} (n + \frac{1}{2})^2}, \quad n = 0, 1, 2, 3, \dots,$$

Since  $k_2^2$  has a negative imaginary part, the poles of concern will be to the right of the negative imaginary axis.

The customary technique for dealing with these poles is to restrict the contours to a set of circular arcs or rectangles such that the contours always pass between the poles. The hyperbolic tangent is analytic everywhere except at such poles, so it is bounded on this set of contours.

Therefore as  $|\lambda| \rightarrow \infty$ , on the appropriate set of contours,

$$Z_2 \sim A_2 \lambda$$

where  $A_2$  is a constant. Then

$$\frac{K_1 - Z_2}{K_1 + Z_2} \rightarrow \frac{A_1 - A_2}{A_1 + A_2} \quad (2.2-5)$$

The asymptotic behavior of the Hankel functions is:

$$H_0^1(\lambda_\rho) \sim \sqrt{\frac{2}{\pi \rho \lambda}} e^{i(\lambda_\rho - \pi/4)} \quad (2.2-6a)$$

$$H_0^2(\lambda_\rho) \sim \sqrt{\frac{2}{\pi \rho \lambda}} e^{-i(\lambda_\rho - \pi/4)} \quad (2.2-6b)$$

of which the first is valid when  $-\pi < \arg(\lambda_\rho) < 2\pi$ , and the second when  $-2\pi < \arg(\lambda_\rho) < \pi$ .

This behavior of the Hankel functions ensures that if the imaginary part of  $\lambda$  is positive, the integrand of  $I_1$  vanishes faster than  $\frac{1}{|\lambda|}$  as  $|\lambda| \rightarrow \infty$ . If the imaginary part of  $\lambda$  is negative, the integrand of  $I_2$  vanishes faster than  $\frac{1}{|\lambda|}$  as  $|\lambda| \rightarrow \infty$ , on the appropriate contours. We will therefore transform  $I_1$  into an integral in the first quadrant and  $I_2$  into an integral in the fourth quadrant.

It is safe to assume, without going into delicacies of mathematical proof, that no poles of the integrand can occur in the first quadrant. If such poles ( $\lambda_n$ ) did occur, they would

give rise to modes of propagation for which, at large distances, the field expressions would contain exponentials of the type  $e^{i\lambda \rho}$ . Since the time factor is  $e^{i\omega t}$ , such expressions would represent fields traveling in the negative  $\rho$  direction, that is, waves converging onto the source. The radiation condition precludes the possibility of such waves.

We use this assumption and the estimates of the integral  $I_1$  as  $|\lambda| \rightarrow \infty$  to deform  $I_1$  into the path

$$I_1 = \frac{M}{2} \int_0^{i\infty} H_0^1(\lambda \rho) G(\beta_1, \beta_2, \beta_3) \lambda d\lambda. \quad (2.2-7)$$

Using a real parameter of integration  $u$ , with  $\lambda = iu$ ,

$$I_1 = -\frac{M}{2} \int_0^{\infty} H_0^1(iu\rho) G(\beta_1, \beta_2, \beta_3) u du \quad (2.2-8a)$$

$$= \left(\frac{M}{2}\right) \frac{2i}{\pi} \int_0^{\infty} K_0(u\rho) G(\beta_1, \beta_2, \beta_3) u du \quad (2.2-8b)$$

where  $K_0(u\rho)$  is defined in the conventional way:

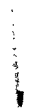
$$K_0(x) = \frac{i\pi}{2} H_0^1(ix).$$

In the fourth quadrant we expect to find poles and branch points. The branch points arise from the quantities  $\beta_n = \sqrt{k_n^2 - \lambda^2}$ .

Let us now treat the  $I_2$  portion of (2.2-3):

$$I_2 = \frac{M}{2} \int_0^{\infty} H_0^2(\lambda \rho) G(\beta_1, \beta_2, \beta_3) \lambda d\lambda.$$

In Figure 2.2 is a representation of the complex  $\lambda$ -plane, on which branch cuts have been drawn. It is convenient in this case to let them be vertical downward; if any of these branch lines happens to cross a pole, it is considered to be deformed slightly to avoid the pole.



We have already verified that integration along the appropriate infinite quarter circle, or portions thereof, contributes nothing. The determination of  $I_2$  is thereby reduced to an integration along the negative imaginary axis, integrations looping back around each branch cut, and an evaluation of the residues at any poles. Thus,

$$I_2 = \left\{ \frac{M}{2} \int_0^{-i\infty} + \frac{M}{2} \int_{\text{B.L.1}} + \frac{M}{2} \int_{\text{B.L.2}} + \frac{M}{2} \int_{\text{B.L.3}} - \left(\frac{M}{2}\right) 2\pi i \sum \text{Res.} \right\} \quad (2.2-9)$$

The first integral in the brackets of (2.2-9) is:

$$\begin{aligned} \frac{M}{2} \int_0^{-i\infty} H_0^2(\lambda_\rho) G(\beta_1, \beta_2, \beta_3) \lambda d\lambda &= -\frac{M}{2} \int_0^\infty H_0^2(-iu_\rho) G(\beta_1, \beta_2, \beta_3) u du \\ &= -\left(\frac{M}{2}\right) \frac{2i}{\pi} \int_0^\infty K_0(u_\rho) G(\beta_1, \beta_2, \beta_3) u du \end{aligned} \quad (2.2-10)$$

This will exactly cancel  $I_1$ , as given by (2.2-8b).

We note now by reference to (2.1-33) and the definition of  $Z_2$  that the integrand is single-valued on the branch cut for  $k_2$ . The integration upward along the left side of this branch cut consequently cancels the integration downward along the other side, and the problem, in its entirety, is further reduced to integration along branch cuts 2 and 3, and the sum of residues.

We will now list the formal expressions for the integrals along branch lines 2 and 3. The discussion of evaluating these is postponed until the next section, in which the magnitudes of the various quantities involved are taken into account and suitable approximations made. Without such approximations it is

more or less impossible to deal with the integrals in a general way.

$$\begin{aligned}
 I_{BL2} &= \frac{M}{2} \int_{k_3-i\infty}^{k_3} H_0^2(\lambda_p) G(\beta_1, \beta_2, \beta_3) \lambda d\lambda + \frac{M}{2} \int_{k_3}^{k_3-i\infty} H_0^2(\lambda_p) G(\beta_1, \beta_2, \beta_3) \lambda d\lambda \\
 &= \frac{M}{2} \int_{k_3-i\infty}^{k_3} H_0^2(\lambda_p) [G(\beta_1, \beta_2, \beta_3) - G(\beta_1, \beta_2, \beta_3)] \lambda d\lambda \quad (2.2-11)
 \end{aligned}$$

where  $\beta_3$  is  $\sqrt{k_3^2 - \lambda^2}$  evaluated as one approaches the branch cut from the left. Likewise,

$$I_{BL3} = \frac{M}{2} \int_{k_1-i\infty}^{k_1} H_0^2(\lambda_p) [G(\beta_1, \beta_2, \beta_3) - G(-\beta_1, \beta_2, \beta_3)] \lambda d\lambda \quad (2.2-12)$$

where  $\beta_1$  is  $\sqrt{k_1^2 - \lambda^2}$  evaluated as one approaches this branch cut from the left.

It remains to specify the contribution from residues at poles. Since the evaluation is at  $(K_1 + Z_2) = 0$ , the integrand concerned becomes

$$H_0^2(\lambda_p) \frac{K_1 - Z_2}{K_1 + Z_2} \frac{e^{i\beta_1(z-z_0)}}{i\beta_1} \lambda \quad (2.2-13)$$

If the poles are designated by  $\lambda_n$ , the formal expression for the contribution from the residues is now (assuming the poles are simple):

$$-2\pi i \left(\frac{M}{2}\right) \sum \text{Res} = -\frac{2i\pi M}{\sigma_1 + i\omega\epsilon_1} \sum H_0^2(\lambda_{np}) \lambda_n \frac{e^{i\beta_1^n(z-z_0)}}{(K_1 + Z_2)_{\lambda=\lambda_n}} \quad (2.2-14)$$

where  $\beta_1^n = \sqrt{k_1^2 - \lambda_n^2}$ . The  $\lambda_n$  are the zeroes of  $K_1 + Z_2$ . The matter of finding these zeroes is deferred until the next section, where specialization to the water-air-ionosphere problem is made.

### 2.3 The Hertz Potential in Region 2.

The same procedure is followed for the development in this case. The integral concerned is (36). In examining the behavior of this integrand for large  $|\lambda|$ , we note that the bracketed term approaches

$$\frac{A_2 \lambda \cos(-i\lambda z) - iA_2 \lambda \sin(-i\lambda z)}{A_1 \lambda + A_2 \lambda} \sim \frac{A_2 e^{-\lambda z}}{A_1 + A_2}$$

where  $A_1$  and  $A_2$  are constants. For  $R_e[\lambda] > 0$  and  $z \geq 0$  this term is evidently bounded along an infinite radius in the right half plane, and the exponential behavior of the Hankel functions ensures that the integrands vanish faster than  $1/|\lambda|$  as  $|\lambda| \rightarrow \infty$ , as before, along the appropriate contours.

As before we will be left with a representation involving integration along the branch cuts and residues at any poles. Again the integrand is single valued around the branch cut for  $\beta_2$  so that the branch line 1 integration gives zero. The branch line 2 integration can be written:

$$I_{BL2} = -iM \int_{k_3 - i\infty}^{k_3} H_0^2(\lambda \rho) [F(\beta_1, \beta_2, \beta_3) - F(\beta_1, \beta_2, \tau\beta_3)] \lambda d\lambda \quad (2.3-1)$$

where

$$F(\beta_1, \beta_2, \beta_3) = \frac{e^{-i\beta_1 z_0}}{\beta_2} \frac{K_2 \cos \beta_2 z - iZ_2 \sin \beta_2 z}{K_1 + Z_2} \quad (2.3-2)$$

The branch line 3 integration is represented by

$$I_{BL3} = -iM \int_{k_1 - i\infty}^{k_1} H_0^2(\lambda\rho) [F(\beta_1, \beta_2, \beta_3) - F(-\beta_1, \beta_2, \beta_3)] \lambda d\lambda \quad (2.3-3)$$

The contribution from residues at poles is given by

$$2\pi M \sum_{\lambda=\lambda_n} H_0^2(\lambda_n\rho) \lambda_n \frac{e^{-i\beta_1^n z_0}}{\beta_2^n} \left\{ \frac{K_2 \cos \beta_2 z - i K_1 \sin \beta_2 z}{(K_1 + Z_2)} \right\} \quad (2.3-4)$$

where obviously the  $\lambda_n$  are the same as for the corresponding expression in region 1.

#### The Hertz Potential in Region 3

Since for our problem this is of relatively less importance than the regions 1 and 2 potentials, we will hereafter omit it from our discussion.

#### 2.4 Specialization to the Water-Air-Ionosphere Problem

In the 1-1000 cps range of frequency  $k_1^2$  is very accurately given by the term  $(-i\omega\mu\sigma_1)$ . This is equivalent to saying that the displacement current in the wave equation is negligibly small compared with the conduction current. The criterion that this be true is that  $\sigma/\omega\epsilon$  be much greater than 1.

The situation is not so clear-cut in the case of the ionospheric propagation constant. For one thing, there is considerable doubt as to the proper value of conductivity to be used. In this paper we are assuming a physical model which represents an extremely simplified picture of the actual situation. The ionosphere is not homogeneous and not sharply bounded; moreover, the bottom portion of its lowest ionized region has

diurnal and seasonal variations. Under such circumstances, the most that can be expected of our theoretical development is that it gives answers which are in reasonably good correspondence with observed phenomena. The "correct" values to be used for  $\sigma$  and  $h$  are those which give results conforming most closely with observation. If the discrepancies are too great, then another, perhaps more elaborate model must be used.

The effective conductivity (the conductivity which, for the model of a sharply bounded homogeneous ionosphere, gives results that compare favorably with experiment), can be estimated from the empirically developed Austin-Cohen formula<sup>6</sup> which gives the attenuation in field strength from one point to another when transmitter and receiver are located on the earth's surface. Such a comparison is made by Bremmer<sup>7</sup>, who then concludes that a conductivity of around  $4 \times 10^{-4}$  mhos per meter is appropriate. The value of  $10^{-4}$  is also used by Schumann.<sup>8</sup> Recent studies seem to show, however, that this value is too great. A large number of "atmospherics", the electromagnetic waves radiated by lightning discharges, were studied over a period of several years by Chapman and Edwards,<sup>9</sup> Chapman and Matthews,<sup>10</sup>

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<sup>6</sup>Austin and Cohen, Bulletin of the Bureau of Standards, 7, 315 (1911)

<sup>7</sup>H. Bremmer, Terrestrial Radio Waves (Elsevier Publishing Company, Amsterdam 1949) p. 226

<sup>8</sup>W. O. Schumann, "Über die oberfelder bei der Ausbreitung langer, elektrischer Wellen" Zeitschrift für angewandte Physik 6, 35 (1959)

<sup>9</sup>F.W. Chapman and A.G. Edwards, Proc. U.R.S.I. 8, pt.2, 351, (1950)

<sup>10</sup>F.W. Chapman and W.D. Matthews, Nature 172, 495 (1953)

Chapman and Jolley<sup>11</sup> and Chapman and Macario.<sup>12</sup> Spectral analyses were performed over the range 40 cps to 16 kc/sec, and marked differences were found, below 10 kc/sec, from the attenuation to be expected on the basis of the Austin-Cohen formula. The whole matter is discussed by Wait,<sup>13</sup> who finds that good agreement is obtained between theoretical and experimental results for VLF if the ionospheric conductivity is taken to be about one two-hundredth of the value used by Bremmer, i.e.,  $\sigma_3 \approx 2 \times 10^{-6}$  mhos/meter. In the ELF range there are indications that penetration of electromagnetic energy is deeper into the ionosphere, so that the effective conductivity as well as the effective height of reflection is greater than the indicated value for VLF. A recent paper by Wait<sup>1</sup> discusses this matter.

In the frequency range 100 cycles/sec to 1000 cycles/sec there is some indication that the attenuation rates are given more exactly when a more elaborate model of the ionosphere is used in the theoretical development. An analysis is made by Wait<sup>14</sup> for an atmosphere-ionosphere model wherein the refractive index is given as a function of height by:

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<sup>11</sup>F. W. Chapman and A. Jolley, Proc. U.R.S.I. 10, part 2, 355 (1950)

<sup>12</sup>F. W. Chapman and R.C.V. Macario, Nature 177, 931, (1956)

<sup>13</sup>J. R. Wait and A. Murphy, Proc. I.R.E. 45, 754 (1957)

<sup>14</sup>J. R. Wait, Journal of Geophys. Research 65, 597 (1960)

$$N(z) = \begin{cases} 1 & 0 < z < h \\ \bar{N} e^{z-h/L} & h < z \end{cases} \quad (2.4-1)$$

In these equations  $h$  is the height at which the refractive index changes discontinuously from 1.0 to some value  $\bar{N}$ , and  $L$  is a scale factor. Best agreement with the results of Chapman and Macario<sup>12</sup> seems to be obtained when  $L$  is chosen as being equal to 30. The result is that the attenuation in nepers per unit length does not vary as  $f^{1/2}$ , as it would for a homogeneous atmosphere, but as  $f^\gamma$ ,  $1/2 < \gamma < 1$ .

The experimental data in the ELF range is open to a certain amount of question, however, and for the submarine problem it would seem most logical to proceed with an analysis based on the homogeneous ionosphere. If necessary, a slight modification may possibly be made in the final answer to account for, at least, the gross effects of an inhomogeneous ionosphere. We will take the height of the lower boundary of the ionosphere as  $9 \times 10^4$  meters, and a conductivity of  $10^{-5}$  mhos/meter, these figures corresponding to ones which have been used by Wait in ELF studies.<sup>1,13</sup>

To go back, then, to the criterion  $\sigma/\omega\epsilon \gg 1$ , we see that the conduction current is much greater than the displacement current provided  $f \ll 1.8 \times 10^{10}$   $\sigma$ . This is quite well satisfied for the ELF range, and will be assumed throughout the remainder of our discussion.

In air, the conduction current is negligibly small compared with the displacement current.<sup>15</sup> For the three regions, then:

$$\begin{aligned} k_1^2 &\approx i\omega\mu_0\sigma_1 \\ k_2^2 &\approx \omega^2\mu_0\epsilon_0 \\ k_3^2 &\approx -i\omega\mu_0\sigma_3. \end{aligned} \quad (2.4-2)$$

Therefore

$$\begin{aligned} K_1(\lambda) &\approx \frac{i\beta_1}{\sigma_1} \\ K_2(\lambda) &\approx \frac{\beta_2}{\omega\epsilon_0} \\ K_3(\lambda) &\approx \frac{i\beta_3}{\sigma_3} \end{aligned} \quad (2.4-3)$$

## 2.5 The Hertz Potential in Water

The branch line 2 integral for region 1 will now be investigated for the water-air-ionosphere problem. From the results of the previous section and using (2.1-28) or (2.1-29) one gets

$$\begin{aligned} B_1 &= \frac{\alpha_1(\alpha_2 - \alpha_3 T) - \alpha_2(\alpha_3 + \alpha_2 T)}{\alpha_1(\alpha_2 - \alpha_3 T) + \alpha_2(\alpha_3 + \alpha_2 T)} \\ &= \frac{\alpha_2(\alpha_1 - \alpha_2 T) - \alpha_3(\alpha_2 + \alpha_1 T)}{\alpha_2(\alpha_1 + \alpha_2 T) + \alpha_3(\alpha_2 - \alpha_1 T)} \end{aligned} \quad (2.5-1)$$

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<sup>15</sup>Although  $k_2^2$  is practically real, it is nevertheless important in arguing the validity of the original formulation of our solutions that it always be understood to retain a small negative imaginary part; hence it is not on the real axis of  $\lambda$ . This matter is discussed thoroughly by G. M. Wing, Mathematical Aspects of the Problem of Acoustic Waves in a Plane Stratified Medium, Sandia Corporation Memorandum, 1960.

where

$$\alpha_1 = \frac{\beta_1}{\sigma_1}, \quad \alpha_2 = \frac{\beta_2}{\omega \epsilon_0}, \quad \alpha_3 = \frac{\beta_3}{\sigma_3}, \quad T = \tan \beta_2 h, \quad (2.5-2)$$

and  $B_1$  has its meaning as previously specified. By reference to equations (2.1-33) and (2.2-1) we see that

$$G(\beta_1, \beta_2, \beta_3) - G(\beta_1, \beta_2, -\beta_3) = \frac{e^{i\beta_1(z-z_0)}}{i\beta_1} [B_1(\alpha_1, \alpha_2, \alpha_3) - B(\alpha_1, \alpha_2, \alpha_3)] \quad (2.5-3)$$

The bracketed part of this can be written

$$\frac{Q - \alpha_3 P}{R + \alpha_3 S} - \frac{Q + \alpha_3 P}{R - \alpha_3 S} = - \frac{2\alpha_3(SQ + PR)}{R^2 - \alpha_3^2 S^2},$$

where

$$Q = \alpha_2(\alpha_1 - \alpha_2 T)$$

$$P = \alpha_2 + \alpha_1 T$$

$$R = \alpha_2(\alpha_1 + \alpha_2 T)$$

$$S = \alpha_2 - \alpha_1 T.$$

This becomes

$$\frac{-4\alpha_1\alpha_2^2\alpha_3}{\alpha_2^2(\alpha_1 C + \alpha_2 S)^2 - \alpha_3^2(\alpha_2 C - \alpha_1 S)^2}$$

where  $S = \sin \beta_2 h$ ,  $C = \cos \beta_2 h$ . It will be convenient to rewrite this as

$$- \frac{4\delta_1\delta_3\beta_1\beta_2^2\beta_3}{D(\lambda)} \quad (2.5-4)$$

where  $\delta_1 = \frac{\sigma_1}{\omega \epsilon_0}$ ,  $\delta_3 = \frac{\sigma_3}{\omega \epsilon_0}$ , and

$$D(\lambda) = \delta_3^2 \beta_2^2 (\beta_1 C + \delta_1 \beta_2 S)^2 - \beta_3^2 (\beta_1 S - \delta_1 \beta_2 C)^2.$$

The branch line 2 integration is now

$$I_{BL2} = 2iM\delta_1\delta_3 \int_{k_3-i\infty}^{k_3} \frac{\beta_2^2 \beta_3}{D(\lambda)} e^{i\beta_1(z-z_0)} H_0^2(\lambda\rho) \lambda d\lambda \quad (2.5-5)$$

We will designate  $k_3 = a(1-i)$  and change the parameter of integration by writing  $\lambda = a(1-i-it)$ . It will be necessary to use the asymptotic approximation to the Hankel function, (42), which is quite good for large values of the argument and which, at least when the index is zero, is within an order of magnitude of the correct value down to values of the argument only a little larger than one. (See, for example, Jahnke and Emde, Tables of Functions, Dover, 1945). In terms of distance and frequency this requires, roughly,  $\rho\sqrt{f} \geq 1.2 \times 10^5$ .

The branch line integral is then

$$I_{BL2} = -2Ma^{5/2} e^{i\pi/4} \delta_1\delta_3 \sqrt{\frac{2}{\pi\rho}} e^{-ia\rho} e^{-a\rho} \int_0^\infty \sqrt{t} F(t) e^{-a\rho t} dt \quad (2.5-6)$$

where

$$F(t) = \frac{\sqrt{t+2(1+i)} \beta_2^2 (1-i-it)^{1/2} e^{i\beta_1(z-z_0)}}{D[\lambda(t)]}$$

in which the  $\beta$ 's are expressed as functions of  $t$ .  $F(t)$  is analytic at  $t = 0$  and is bounded on the real axis of  $t$ . It follows from Watson's lemma<sup>16</sup> that an asymptotic expansion exists and can be obtained by integrating the terms resulting from a Taylor series expansion of  $F(t)$  around  $t = 0$ .

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<sup>16</sup>Jeffreys and Jeffreys, Methods of Mathematical Physics, (Cambridge University Press, 1946) pp. 471, 668.

It is readily found that

$$F(0) = \frac{2 e^{ib_1(z-z_0)}}{\delta_3^2 [b_1 \cos hb_2 + \delta_1 b_2 \sin hb_2]^2} \quad (2.5-7)$$

where

$$b_1 = \sqrt{k_1^2 + 2ia^2}$$

$$b_2 = \sqrt{k_2^2 + 2ia^2}.$$

Since  $k_1^2 \gg 2ia^2$ , it will be sufficiently accurate to take

$$b_1 = k_1.$$

The first term arising from an integration over the Taylor series expansion of  $F(t)$  is therefore proportional to:

$$\int_0^\infty F(0) \sqrt{t} e^{-a_\rho t} dt = \frac{\pi}{2} F(0) \left( \frac{1}{a_\rho} \right)^{3/2} \quad (2.5-8)$$

If the second term were obtained, it would contain the factor

$$F'(0) \frac{3\sqrt{\pi}}{4} \left( \frac{1}{a_\rho} \right)^{5/2} \quad (2.5-9)$$

and so forth, in increasing inverse powers of  $\rho$ . With multiplication by the factor  $\rho^{-1/2}$  from outside the integral, it can be concluded that the branch line integral is attenuated at least as  $e^{-a_\rho/\rho^2}$  at large distances. Since the factors whose product is  $F(t)$  are smoothly varying near  $t = 0$ , it is expected that  $F'(0)$ ,  $F''(0)$ , etc. will not be excessively large compared with  $F(0)$ , and that as an estimate when  $\rho$  is of the order of 1000 km or greater, it will suffice to consider only this  $\rho^{-2}$  term.

The complete expression for the branch line integral is:

$$I_{BL2} \approx 2\sqrt{2} M e^{-i3\pi/4} \frac{\delta_1 a e^{ik_1(z-z_0)}}{\delta_3 (b_1 \bar{C} + \delta_1 b_2 \bar{S})^2} \frac{e^{-a\rho(1+i)}}{\rho^2} \quad (2.5-10)$$

where we have abbreviated  $\bar{C} = \cos hb_2$  and  $\bar{S} = \sin hb_2$ .

When  $|k_2| \ll |k_3|$ , which requires

$$f \ll 2 \times 10^{10} \sigma_3 \quad (2.5-11)$$

the form of  $F(t)$  can be simplified somewhat. Under this condition  $D(t)$  can be written

$$D(t) \approx \delta_1^2 \delta_3^2 \beta_2^4(t) S^2(t) . \quad (2.5-12)$$

Then (2.5-5) assumes the form

$$\bar{I}_{BL2} \approx \frac{2iM}{\delta_1 \delta_3} \int_{k_3-i\infty}^{k_3} e^{i\beta_1(z-z_0)} \frac{\beta_3}{\beta_2^2 S^2} H_0^2(\lambda \rho) \lambda d\lambda \quad (2.5-13)$$

Substituting  $\lambda = a(1-i-it)$  and using the asymptotic expression for the Hankel function,

$$\bar{I}_{BL2} \approx -2Me^{i\pi/4} \sqrt{\frac{2}{\pi\rho}} \frac{a^{5/2}}{\delta_1 \delta_3} e^{-a\rho(1+i)} \int_0^\infty \sqrt{t} \bar{F}(t) e^{-a\rho t} dt \quad (2.5-14)$$

where

$$\bar{F}(t) = \frac{[t+2(1+i)]^{1/2} (i-i-it)^{1/2}}{\beta_2^2 S^2} e^{i\beta_1(z-z_0)} .$$

Evidently

$$\bar{F}(0) = \frac{2}{b_2^2 \bar{S}^2} e^{ib_1(z-z_0)} .$$

Making the expansion of  $\bar{F}(t)$  in Taylor series therefore gives as the first term estimate of  $\bar{I}_{BL2}$

$$\bar{I}_{BL2} \approx M \frac{2\sqrt{2}e^{-i3\pi/4}a}{\delta_1 \delta_3 b_2^2 \bar{S}^2} e^{ik_1(z-z_0)} \frac{e^{-a\rho(1+i)}}{\rho^2} \quad (2.5-15)$$

The integral around branch cut 3 must now be considered. One can see rather quickly, however, that the contribution from this integral must be quite small at large distances. This is so because of the exponential term arising from the Hankel function. For  $\lambda$  in the vicinity of  $k_1$ , which has an imaginary part equal to about  $3.98 \times 10^{-3} \sqrt{f}$ , and for  $\rho$  around 100 km, obviously the attenuation will be enormous for ELF or higher frequencies. We will therefore neglect this contribution.

The next task is to solve the equation  $K_1 + Z_2 = 0$ , in order that evaluation of the expression (2.2-14) can be made. The roots  $\lambda_n$  cannot be determined generally, and it is necessary to consider our situation in a rather specific way.

From the values for  $K_n(\lambda)$  given by (2.4-3), it can be determined that the zeroes of  $K_1 + Z_2$  are obtainable by solving the equation

$$\frac{\tan h\beta_2}{\beta_2} = \frac{\delta_1\beta_3 + \delta_3\beta_1}{\beta_1\beta_3 - \delta_1\delta_3\beta_2^2}^2 \quad (2.5-16)$$

This is quite a formidable task. It is useless to search for real values of  $\lambda_n$ , since this would imply lossless propagation in the radial direction. In the analysis of a physical problem such propagation will appear to exist only in the case when some non-realizable perfection has been assigned to the media in which propagation occurs, i.e., perfectly reflecting boundaries, and/or perfect -- that is, lossless -- dielectrics. Since no such assumptions have been made in this problem, we immediately resign ourselves to the requirement that the roots

$\lambda_n$  will be complex. This can, in fact, be verified from (2.5-16) without great difficulty.

At this point it is possible to invoke a physical argument. If the earth and ionosphere boundaries were perfectly conducting, only the so-called TEM mode of propagation could have a real propagation constant, for the 1-1000 cps frequency range. This would be exactly  $k_2$ . The propagation constants for other modes, if any were excited, would be purely imaginary and such modes would be exponentially damped to a very high degree. This is because the distance between boundaries is considerably less than one half wave length for any frequency in this range. The introduction of finite conductivity in the walls causes nonzero electric field tangential to the walls but as long as the conductivity can be considered high ( $\sigma/\omega\epsilon \gg 1$ ) the propagation is still considered to be of the TEM type. As the resistivity varies continuously from zero to a small but non-zero value, the associated propagation constant must vary continuously from  $k_2$  to some value in the vicinity of  $k_2$ . If any poles were on the imaginary axis, they would also move continuously to new positions, but their new locations would not be drastically closer to the real axis. This matter is discussed in detail in Appendix A. On the basis of the results developed there, we venture to assert that only one root will be found in the vicinity of the real axis, and that this root will not be too far removed from  $k_2$ . We will therefore use the small argument approximation to the tangent:

$$\tan h\beta_2 \approx h\beta_2 \quad (2.5-17)$$

which is reasonably valid if  $|h\beta_2| \lesssim 0.4$ , implying

$$\left| \sqrt{k_2^2 - \lambda^2} \right| \lesssim 4.43 \times 10^{-6}$$

where an ionosphere height of  $9 \times 10^4$  meters has been assigned.

Letting the root be  $\lambda_0$ , we represent  $\lambda_0^2$  as  $k_2^2 (1 + \Delta)$ ; then

$$\begin{aligned} k_2^2 |\Delta| &\lesssim 1.96 \times 10^{-11} \\ |\Delta| &\lesssim \frac{4.47 \times 10^4}{f^2} \end{aligned} \quad (2.5-18)$$

After  $\Delta$  has been determined, we will be able to estimate the range of validity for the approximation (2.5-17).

We will arbitrarily assign  $n = 0$  to the root to be obtained in this case. Then:

$$h(\beta_1\beta_3 - \delta_1\delta_3\beta_2^2) \approx \delta_1\beta_3 + \delta_3\beta_1. \quad (2.5-19a)$$

We approximate  $\beta_1 \approx k_1$ . Then

$$\delta_1^2(k_2^2 - \lambda_0^2) \approx \delta_3^2 [k_1 + \delta_1 h(k_2^2 - \lambda_0^2)]^2 \quad (2.5-19b)$$

We assume the parameters to be of the order of magnitude of the following:

$$\begin{aligned} h &= 9 \times 10^4 \text{ meters} \\ \sigma_1 &= 4 \text{ mho/meter} \\ \sigma_3 &= 10^{-5} \text{ mho/meter} \\ \delta_1 &= 7.2 \times 10^{10}/f \\ \delta_3 &= 1.8 \times 10^5/f \\ k_1 &= (1-i) 3.98 \times 10^{-3} \sqrt{f} \\ k_3 &= (1-i) 6.28 \times 10^{-6} \sqrt{f} \\ k_2 &= 2.09 \times 10^{-8} f \end{aligned} \quad (2.5-20)$$

The choice of  $h$  and  $\sigma_3$  has already been discussed. The value for  $\sigma_1$  is commonly accepted as a convenient average for sea water, although in certain cases it might have to be revised downward or upward, as for example near large quantities of polar ice in the first instance, or in brackish seas with high rates of evaporation in the second. Any minor adjustments will not affect the validity of the following equations.

Then:

$$\lambda_o^2 \approx k_2^2 \left[ 1 + \frac{1-i}{h \sqrt{2\omega\mu\sigma_3}} \right] \quad (2.5-21)$$

From this,  $|\Delta| = \frac{1}{h \sqrt{\omega\mu\sigma_3}}$ . For the values of the parameters exactly as above, this is given numerically by  $\frac{1.25}{f}$ . From (2.5-18) we find:

$$f \lesssim 2340 \text{ cps}$$

which validates (2.5-17) well over the entire 1-1000 cps range.

A final expression for  $\lambda_o$  can be written

$$\lambda_o = k_2 \left[ 1 + (1-i) \frac{252}{h \sqrt{f\sigma_3}} \right]^{1/2} \quad (2.5-22a)$$

where, of course, the square root in the fourth quadrant is to be taken. With the parameters listed above, this becomes

$$\lambda_o = k_2 \left[ 1 + (1-i) \frac{0.88}{\sqrt{f}} \right]^{1/2} \quad (2.5-22b)$$

We see from (2.5-22a) that the root approaches  $k_2$  at higher frequencies. We also see that as  $\sigma_3 \rightarrow \infty$ ,  $\lambda_o \rightarrow k_2$ . Actually, of course, if we let the ionosphere be a perfect reflector and the earth an imperfect one,  $\lambda_o$  should assume a value very close

to  $k_2$  but with a small imaginary part that takes into account losses in the sea. In our equation this term does not appear, since it has been discarded as negligible in comparison with the term arising from losses in the ionosphere.

For convenience we here summarize the expressions for Hertz potential in water due to the vertical electric dipole:

Branch line 2:

$$M \frac{2\sqrt{2}}{\delta_1 \delta_3} \frac{e^{-i3\pi/4} a}{b_2^2 s^2} e^{ik_1(z-z_0)} \frac{e^{-a\rho(1+l)}}{\rho^2} \quad (2.5-15)$$

Mode:

$$- \frac{2i\pi M}{\sigma_1} \lambda_0 H_0^2 (\lambda_0 \rho) \frac{e^{i\beta_1^0(z-z_0)}}{(k_1 + z_2)^2} \Big|_{\lambda=\lambda_0} \quad (2.5-23)$$

in which  $\beta_1^0 = \sqrt{k_1^2 - \lambda_0^2}$ .

## 2.6 The Hertz Potential in Air:

From equations (2.3-1) and (2.3-2) we not take up the branch line 2 integral in a detailed way. We find

$$F(\beta_1, \beta_2, \beta_3) - F(\beta_1, \beta_2, -\beta_3) =$$

$$\frac{e^{-i\beta_1 z_0}}{\beta_2} \left\{ \cos \beta_2 z \left[ \frac{K_2}{K_1 + z_2(\beta_1, \beta_2, \beta_3)} - \frac{K_2}{K_1 + z_2(\beta_1, \beta_2, -\beta_3)} \right] - i \sin \beta_2 z \left[ \frac{z_2(\beta_1, \beta_2, \beta_3)}{K_1 + z_2(\beta_1, \beta_2, \beta_3)} - \frac{z_2(\beta_1, \beta_2, -\beta_3)}{K_1 + z_2(\beta_1, \beta_2, -\beta_3)} \right] \right\} \quad (2.6-1)$$

Since

$$\frac{K_2}{K_1 + z_2} = \frac{-i\delta_1\beta_2(\delta_3\beta_2^C - \beta_3^S)}{\beta_1(\delta_3\beta_2^C - \beta_3^S) + \delta_1\beta_2(\beta_3^C + \delta_3\beta_2^S)} \quad (2.6-2)$$

it is found that the multiplier of  $\cos \beta_2 z$  is given by

$$\frac{2i\delta_1^2 \delta_3 \beta_2^3 \beta_3}{P(\lambda)} \quad (2.6-3)$$

where

$$P(\lambda) = \delta_3^2 \beta_2^2 (\beta_1 C + \delta_1 \beta_2 S)^2 - \beta_3^2 (\beta_1 S - \delta_1 \beta_2 C)^2$$

The term multiplying  $\sin \beta_2 z$  is given by

$$\begin{aligned} \delta_1 \beta_2 \left[ \frac{\beta_3 + \delta_3 \beta_2 T}{\delta_3 \beta_2 (\beta_1 + \delta_1 \beta_2 T) + \beta_3 (\delta_1 \beta_2 - \beta_1 T)} - \frac{\beta_3 - \delta_3 \beta_2 T}{\delta_3 \beta_2 (\beta_1 + \delta_1 \beta_2 T) - \beta_3 (\delta_1 \beta_2 - \beta_1 T)} \right] \\ = \frac{2 \delta_1 \delta_3 \beta_1 \beta_2^2 \beta_3}{P(\lambda)} \end{aligned} \quad (2.6-4)$$

Expression (2.6-1) becomes:

$$\frac{e^{-i\beta_1 z_0}}{i\beta_2} \left\{ \frac{2\delta_1 \delta_3 \beta_2^2 \beta_3 (\beta_1 \sin \beta_2 z - \delta_1 \beta_2 \cos \beta_2 z)}{P(\lambda)} \right\}. \quad (2.6-5)$$

The integral is then

$$-2M\delta_1 \delta_3 \int_{k_3 - i\infty}^{k_3} H_0^2(\lambda \rho) e^{-i\beta_1 z_0} \beta_2 \beta_3 \left\{ \frac{\beta_1 \sin \beta_2 z - \delta_1 \beta_2 \cos \beta_2 z}{P(\lambda)} \right\} \lambda d\lambda \quad (2.6-6)$$

When the change of variable  $\lambda = a(1-i-it)$  is made and the asymptotic form of the Hankel function used, this becomes

$$-i2M\delta_1 \delta_3 \sqrt{\frac{2}{\pi \rho}} e^{-a\rho(1+i)} a^{5/2} \int_0^\infty \sqrt{t} R(t, z) e^{-a\rho t} dt \quad (2.6-7)$$

where

$$R(t, z) = \beta_2 [t+2(1+i)]^{1/2} (1-i-it)^{1/2} \left( \frac{\beta_1 \sin \beta_2 z - \delta_1 \beta_2 \cos \beta_2 z}{P(t)} \right) e^{-i\beta_1 z_0}$$

Then

$$R(0, z) = \frac{2(b_1 \sin b_2 z - \delta_1 b_2 \cos b_2 z)}{\delta_3^2 b_2 (b_1 \bar{C} + \delta_1 b_2 \bar{S})^2} e^{-ib_1 z_0} \quad (2.6-8)$$

Therefore, at sufficiently large distances, this branch line integral behaves as

$$\frac{-i2\sqrt{2} M a e^{-ik_1 z_0}}{\delta_1 \delta_3 b_2 \bar{S}^2} \frac{e^{-a\rho(1+i)}}{\rho^2} (b_1 \sin b_2 z - \delta_1 b_2 \cos b_2 z) \quad (2.6-9)$$

The mode contribution follows immediately from (2.3-4):

$$\frac{2\pi M \lambda_0}{\sigma_1} H_0^2(\lambda_0 \rho) e^{-i\beta_1^0 z_0} \frac{\delta_1 \cos \beta_2^0 z + \left( \frac{\beta_1^0}{\beta_2^0} \right) \sin \beta_2^0 z}{(K_1 + Z_2)' \lambda = \lambda_0} \quad (2.6-10)$$

### 3.0 THE SUBMERGED HORIZONTAL ELECTRIC DIPOLE

#### 3.1 Introduction of Equations

Although the extension of the previous results to the case of the horizontal electric dipole is straightforward, it is considerably more involved algebraically. The Hertz vector must be written with two components: one horizontal, the other vertical, in order that the boundary conditions be satisfied.<sup>17,4</sup>

We will take the dipole orientation to be along the x-axis. Then the two components of the Hertz vector are  $\Pi_x$  and  $\Pi_z$ . The field components are derived in the usual manner:

$$\begin{aligned}\vec{E}_n &= k_n^2 \vec{\Pi}_n + \nabla \nabla \cdot \vec{\Pi}_n \\ \vec{H}_n &= \frac{k_n^2}{i\omega\mu_0} \nabla \times \vec{\Pi}_n\end{aligned}\quad n = 1, 2, 3 \quad (3.1-1)$$

To match boundary conditions we will need:

$$E_{xn} = k_n^2 \Pi_{xn} + \frac{\partial}{\partial x} \nabla \cdot \vec{\Pi}_n \quad n = 1, 2, 3 \quad (3.1-2)$$

$$E_{yn} = \frac{\partial}{\partial y} \nabla \cdot \vec{\Pi}_n \quad n = 1, 2, 3 \quad (3.1-3)$$

$$H_{xn} = \frac{k_n^2}{i\omega\mu_0} \frac{\partial \Pi_{zn}}{\partial y} \quad n = 1, 2, 3 \quad (3.1-4)$$

$$H_{yn} = \frac{k_n^2}{i\omega\mu_0} \left( \frac{\partial \Pi_{xn}}{\partial z} - \frac{\partial \Pi_{zn}}{\partial x} \right) \quad n = 1, 2, 3 \quad (3.1-5)$$

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<sup>17</sup>A. Sommerfeld, Partial Differential Equations, (Academic Press Inc., New York, 1949) p. 258.

These will be continuous across the interface if we require, from (3.1-2) and (3.1-3):

$$\nabla \cdot \vec{\Pi}_n = \nabla \cdot \vec{\Pi}_{n+1} \quad n=1, z=0 \quad (3.1-6)$$

$$k_n^2 \Pi_{xn} = k_{n+1}^2 \Pi_{x(n+1)} \quad n=2, z=h \quad (3.1-7)$$

From (3.1-4) we require

$$k_n^2 \Pi_{zn} = k_{n+1}^2 \Pi_{z(n+1)} \quad \begin{matrix} n=1, z=0 \\ n=2, z=h \end{matrix} \quad (3.1-8)$$

and from (3.1-5),

$$k_n^2 \frac{\partial \Pi_{xn}}{\partial z} = k_{n+1}^2 \frac{\partial \Pi_{x(n+1)}}{\partial z} \quad (3.1-9)$$

We will first develop the formulae for  $\Pi_x$ . In this case, as for the vertical dipole, there is primary stimulation in medium 1 only, but secondary stimulation in all media. In the same way as for equation (2.1-18) we write:

$$\Pi_{x_1(\text{pri})} = M \int_0^\infty \frac{\lambda}{i\beta_1} J_0(\lambda\rho) e^{-i\beta_1 |z+z_0|} d\lambda \quad (3.1-10)$$

$$\Pi_{x_1(\text{sec})} = M \int_0^\infty w_1(\lambda) e^{i\beta_1 z} \frac{\lambda e^{-i\beta_1 z_0}}{i\beta_1} J_0(\lambda) d\lambda \quad (3.1-11)$$

so that the total Hertzian vector x-component in medium 1 is

$$\Pi_{x_1} = M \int_0^\infty \left[ e^{-i\beta_1 |z+z_0|} + w_1(\lambda) e^{i\beta_1 (z-z_0)} \right] \frac{\lambda}{i\beta_1} J_0(\lambda\rho) d\lambda \quad (3.1-12)$$

In the other media:

$$\Pi_{x_2} = M \int_0^\infty \left[ v_2(\lambda) e^{-i\beta_2 z} + w_2(\lambda) e^{i\beta_2 z} \right] \frac{\lambda}{i\beta_1} e^{-i\beta_1 z_0} J_0(\lambda\rho) d\lambda \quad (3.1-13)$$

$$\Pi_{x_3} = M \int_0^{\infty} v_3(\lambda) e^{-i\beta_3 z} \frac{\lambda}{i\beta_1} e^{-i\beta_1 z_0} J_0(\lambda \rho) d\lambda . \quad (3.1-14)$$

Application of conditions (3.1-7) and (3.1-9) now yields:

$$\begin{aligned} k_1^2 \beta_1 (w_1 - 1) &= k_2^2 \beta_2 (w_2 - v_2) , \\ k_2^2 \beta_2 (w_2 e^{i\beta_2 h} - v_2 e^{-i\beta_2 h}) &= - k_3^2 \beta_3 v_3 e^{-i\beta_3 h} , \\ k_1^2 (w_1 + 1) &= k_2^2 (w_2 + v_2) , \\ k_2^2 (w_2 e^{i\beta_2 h} + v_2 e^{-i\beta_2 h}) &= k_3^2 v_3 e^{-i\beta_3 h} . \end{aligned} \quad (3.1-15)$$

From these we find

$$w_1(\lambda) = \frac{\beta_1(\lambda) - U_2(\lambda)}{\beta_1(\lambda) + U_2(\lambda)} , \quad (3.1-16)$$

where

$$U_2 = \beta_2 \left[ \frac{\beta_3 + i\beta_2 T}{\beta_2 + i\beta_3 T} \right] , \quad T = \tan \beta_2 h , \quad (3.1-17)$$

and

$$v_2(\lambda) = - i\delta_1 \frac{\beta_1}{\beta_2} \left( \frac{\beta_2 + U_2}{\beta_1 + U_2} \right) , \quad (3.1-18)$$

$$w_2(\lambda) = - i\delta_1 \frac{\beta_1}{\beta_2} \frac{\beta_2 - U_2}{\beta_1 + U_2} , \quad (3.1-19)$$

$$v_3(\lambda) = 2 \frac{\delta_1}{\delta_3} \frac{\beta_1 \beta_2 e^{i\beta_3 h}}{(\beta_1 + U_2)(\beta_2 + i\beta_3 s)} . \quad (3.1-20)$$

Evidently  $U_2$  is even in  $\beta_2$ , hence  $w_1$  and therefore  $\Pi_{x_1}$  are even in  $\beta_2$ .  $\Pi_{x_2}$  can be written in the form

$$\Pi_{x_2} = 2M \frac{k_1^2}{k_2^2} \int_0^\infty \left( \frac{\beta_2 \cos \beta_2 z - iU_2 \sin \beta_2 z}{\beta_1 + U_2} \right) \frac{e^{-i\beta_1 z_0}}{i\beta_2} J_0(\lambda \rho) \lambda d\lambda \quad (3.1-21)$$

from which we see that it, too, is even in  $\beta_2$ .

We will now determine the form of  $\Pi_z$ . From (3.1-6) we note that at the boundaries  $z = 0$  and  $h$ ,

$$\frac{\partial \Pi_{z_n}}{\partial z} - \frac{\partial \Pi_{z(n+1)}}{\partial z} = \frac{\partial \Pi_{x(n+1)}}{\partial x} - \frac{\partial \Pi_{x_n}}{\partial x} \quad (3.1-22)$$

for  $n = 1$  and  $2$ , respectively. However

$$\frac{\partial \Pi_{x_n}}{\partial x} = \frac{\partial \Pi_{x_n}}{\partial \rho} \frac{\partial \rho}{\partial x} = \cos \varphi \frac{\partial \Pi_{x_n}}{\partial \rho} \quad (3.1-23)$$

Therefore a factor  $\cos \varphi$  must occur in the left side of equation (3.1-22), and apparently we will not be able to use functions of the type  $J_0(\lambda \rho) e^{\pm i\beta_n z}$ , but must resort to those of index 1:

$$\cos \varphi J_1(\lambda \rho) e^{\pm i\beta_n z} \quad (3.1-24)$$

No primary stimulation is present in this case. Then in the three media we can write:

$$\Pi_{z_1} = M \cos \varphi \int_0^\infty \psi_1(\lambda) e^{i\beta_1 z} J_1(\lambda \rho) \frac{e^{-i\beta_1 z_0}}{i\beta_1} \lambda^2 d\lambda \quad (3.1-25)$$

$$\Pi_{z_2} = M \cos \varphi \int_0^\infty [\psi_2(\lambda) e^{-i\beta_2 z} + \varphi_2(\lambda) e^{i\beta_2 z}] J_1(\lambda \rho) \frac{e^{-i\beta_1 z_0}}{i\beta_1} \lambda^2 d\lambda \quad (3.1-26)$$

$$\Pi_{z_3} = M \cos \varphi \int_0^\infty \psi_3(\lambda) e^{-i\beta_3 z} J_1(\lambda \rho) \frac{e^{-i\beta_1 z_0}}{i\beta_1} \lambda^2 d\lambda \quad (3.1-27)$$

We have modified the coefficient in each case by inclusion of the factor  $\lambda^2 e^{-i\beta_1 z_0} / i\beta_1$ . This will simplify, somewhat, the determination of the coefficients from the boundary conditions.

We must satisfy at  $z=0$  and  $z=h$ :

$$k_n^2 \Pi_{z_n} = k_{n+1}^2 \Pi_{z(n+1)} \quad (3.1-8)$$

$$\frac{\partial \Pi_{z_n}}{\partial z} - \frac{\partial \Pi_{z(n+1)}}{\partial z} = \cos \varphi \left( \frac{\partial \Pi_{x(n+1)}}{\partial \rho} - \frac{\partial \Pi_{x_n}}{\partial \rho} \right) \quad (3.1-28)$$

for  $n = 1$  and  $2$ , respectively. Four equations are obtained, as follows:

$$k_1^2 \varphi_1 = k_2^2 (\psi_2 + \varphi_2) \quad ,$$

$$k_2^2 (\psi_2 e^{-i\beta_2 h} + \varphi_2 e^{i\beta_2 h}) = k_3^2 \psi_3 e^{-i\beta_3 h} \quad , \quad (3.1-29)$$

$$i\beta_2 \psi_2 - i\beta_2 \varphi_2 + i\beta_1 \varphi_1 = \frac{2\beta_1}{\beta_1 + u_2} (1 + i\delta_1) \quad ,$$

$$-i\beta_2 \psi_2 e^{-i\beta_2 h} + i\beta_2 \varphi_2 e^{i\beta_2 h} + i\beta_3 \psi_3 e^{-i\beta_3 h} = -2 \frac{\delta_1}{\delta_3} \frac{(i\delta_3 + 1) \beta_1 \beta_2}{(\beta_1 + u_2)(\beta_2 c + i\beta_3 s)}$$

From these we find

$$\varphi_1 = \frac{2\beta_1 [(1-\delta_1)(\beta_3 s - \delta_3 \beta_2 c)(\beta_2 c + i\beta_3 s) - \delta_1(1+\delta_3)\beta_2^2]}{(\beta_2 c + i\beta_3 s)(\delta_3 \beta_2 c - \beta_3 s)(\beta_1 + u_2)(\beta_1 + x_2)} \quad (3.1-30)$$

$$\psi_2 = -\frac{1}{2\beta_2} \left[ (\beta_1 + i\delta_1 \beta_2) \varphi_1 + \frac{2(1-\delta_1)\beta_1}{\beta_1 + u_2} \right] \quad (3.1-31)$$

$$\varphi_2 = \frac{1}{2\beta_2} \left[ (\beta_1 - i\delta_1 \beta_2) \varphi_1 + \frac{2(1-\delta_1)\beta_1}{\beta_1 + u_2} \right] \quad (3.1-32)$$

$$\psi_3 = \frac{e^{i\beta_2 h}}{\delta_3 \beta_2} \left[ (\beta_1 s + \delta_1 \beta_2 c) \varphi_1 + \frac{2(1-\delta_1)\beta_1 s}{\beta_1 + \delta_2} \right] \quad (3.1-33)$$

where  $X_2 = (-i\sigma_1 Z_2)$ , with  $Z_2$  defined as in (2.1-31). We see that  $\varphi_1$  is even in  $\beta_2$ , and it is readily verified that  $(\psi_2 e^{-i\beta_2 z} + \varphi_2 e^{i\beta_2 z})$  is also even in  $\beta_2$ .

### 3.2 The Hertz Potential in Region 1

We first take up the  $\Pi_x$  component, hence we refer to (3.1-12) in connection with (3.1-16). (3.1-12) is changed to a contour integral in the complex  $\lambda$ -plane by precisely the same methods that were used in the case of the vertical electric dipole. The portion of the integrand which multiplies the Bessel function is easily seen to be well behaved as  $|\lambda| \rightarrow \infty$  in the right half plane, when contours are taken between poles of the hyperbolic tangent, as discussed previously. Hence the integrands again vanish faster than  $\frac{1}{|\lambda|}$  on the appropriate portions of the infinite semicircle when  $|\text{Im } \lambda| > 0$ .

We are again concerned with branch points in the fourth quadrant arising from the points where  $\beta_1$ ,  $\beta_2$ , and  $\beta_3$  are zero. The integrand is even in  $\beta_2$ , so that once more the problem reduces to integration along branch cuts for  $\beta_1$  and  $\beta_3$ , plus the determination of residues at any poles that may exist.

The branch cut for  $\beta_3$  (branch line 2) will again be made vertically downward from the point  $k_3$ . Then the integral concerned is

$$I_{BL2} = \frac{M}{2} \int_{k_3-i\infty}^{k_3} [W_1(\beta_1, \beta_2, \beta_3) - W_1(\beta_1, \beta_2, -\beta_3)] e^{i\beta_1(z-z_0)} \frac{\lambda}{i\beta_1} H_0^2(\lambda\rho) d\lambda \quad (3.2-1)$$

Equation (3.1-16) gives the form of  $W_1$ .

As for the case of the vertical dipole, the branch line 3 integration as it occurs in our problem can be expected to be negligible at appreciable distances; consequently we will not need a formal expression for it.

An expression for mode contributions due to the  $\Pi_x$  component of the Hertz vector will also be omitted. It will be found later that the mode represented by the pole  $\lambda_0$  is not excited by this component, and as argued previously and in Appendix A, no other modes of importance are excited.

We next take up the  $\Pi_x$  component of the Hertz potential. The transformation to a contour integral in the complex  $\lambda$ -plane proceeds as before, except that now we deal with Hankel functions of order one. We note that

$$\begin{aligned} \frac{d}{dz} H_0^1(z) &= -H_1^1(z) , \\ \frac{d}{dz} H_0^2(z) &= -H_1^2(z) . \end{aligned} \quad (3.2-2)$$

We showed in the analysis for the vertical electric dipole that

$$\int_0^{i\infty} H_0^1(\lambda\rho) F(\lambda) \lambda d\lambda = i \int_0^{-i\infty} H_0^2(\lambda\rho) F(\lambda) \lambda d\lambda \quad (3.2-3)$$

where  $F(\lambda)$  is any function which is even in  $\lambda$ . We take the partial derivative with respect to  $\rho$ , obtaining:

$$-\int_0^{i\infty} H_1^1(\lambda_0) F(\lambda) \lambda^2 d\lambda = \int_0^{-i\infty} H_1^2(\lambda_0) F(\lambda) \lambda^2 d\lambda, \quad (3.2-4)$$

showing that the integrals along the imaginary axis cancel in this case, also. Since the behavior of these Hankel functions for large argument is

$$\begin{aligned} H_1^1(x) &= \sqrt{\frac{2}{\pi x}} e^{i(x - \frac{3\pi}{4})}, \\ H_1^2(x) &= \sqrt{\frac{2}{\pi x}} e^{-i(x - \frac{3\pi}{4})}, \end{aligned} \quad (3.2-5)$$

we can apply the same reasoning as before to the analysis of the other portions of the contour at radius R, as  $R \rightarrow \infty$ .

The integrand of (3.1-25) is single valued around branch line 1, so the integration around branch line 1 is zero, as before.

The integral around branch cut 2 is readily obtained by reference to (3.1-25). The expression will be, after we have converted to the Hankel function representation:

$$I_{BL2} = \frac{M}{2} \cos \varnothing \int_{k_3 - i\infty}^{k_3} [\varnothing_1(\beta_1, \beta_2, \beta_3) - \varnothing_1(\beta_1, \beta_2, -\beta_3)] e^{i\beta_1 z} H_1^2(\lambda_0) \frac{e^{-i\beta_1 z_0}}{i\beta_1} \lambda^2 d\lambda. \quad (3.2-6)$$

The remaining item of interest is the mode solution.

This can be expressed by:

$$2\pi M \cos \varnothing \sum_n \lambda_n H_1^2(\lambda_n) e^{i\beta_1^n (z - z_0)} [h(\beta_1, \beta_2, \beta_3)]_{\lambda=\lambda_n} \quad (3.2-7)$$

where

$$h(\beta_1, \beta_2, \beta_3) = \frac{(i - \delta_1)(\beta_3 s - \delta_3 \beta_2 c)(\beta_2 c + i\beta_3 s) - \delta_1(i + \delta_3)\beta_2^2}{-\frac{1}{\lambda_n} \frac{\partial}{\partial \lambda} [(\beta_2 c + i\beta_3 s)(\delta_3 \beta_2 c - \beta_3 s)(\beta_1 + u_2)(\beta_1 + x_2)]}$$

### 3.3 The Hertz Potential in Region 2.

Taking up first the  $\Pi_x$  component we note from (3.1-13), (3.1-16), and (3.1-19) that the part of the integrand multiplying the Bessel function is well behaved as  $|\lambda| \rightarrow \infty$ , so that the contour integration can be performed in the same way as before.

The branch line 2 expression is written:

$$I_{BL2} = \frac{M}{2} \int_{k_3 - i\infty}^{k_3} \left\{ e^{-i\beta_2 z} [v_2(\beta_1, \beta_2, \beta_3) - v_2(\beta_1, \beta_2, -\beta_3)] + e^{i\beta_2 z} [w_2(\beta_1, \beta_2, \beta_3) - w_2(\beta_1, \beta_2, -\beta_3)] \right\} \frac{\lambda}{i\beta_1} e^{-i\beta_1 z_0} H_0^2(\lambda \rho) d\lambda. \quad (3.3-1)$$

Further evaluation will be done later.

As mentioned in the last section, the important mode is not excited by the  $\Pi_x$  component, so an expression for contributions due to poles will be omitted.

In the case of the  $\Pi_z$  component we note from (3.1-26), (3.1-31), and 3.1-32), the fact that  $\phi_1(\lambda)$  is well behaved as  $|\lambda| \rightarrow \infty$ , so that our contour integration can proceed as for the region 1 discussion.

The form of the branch line integral for  $\Pi_z$  can be obtained from (3.1-26). Specifically:

$$\Pi_{z2} = \frac{M}{2} \cos \phi \int_C \left( \psi_2 e^{-i\beta_2 z} + \phi_2 e^{i\beta_2 z} \right) H_1^2(\lambda \rho) \frac{e^{-i\beta_1 z_0}}{i\beta_1} \lambda^2 d\lambda, \quad (3.3-2)$$

where C is the contour around the branch cut and  $\psi_2$  and  $\phi_2$  are specified by (3.1-31) and (3.1-32). We will leave it in this form for the present to avoid writing a cumbersome expression.

Also from (3.1-26) we find the expression for the residues at the poles. This is:

$$\pi M \cos \varnothing \sum H_1^2(\lambda_n) \frac{e^{-i\beta_1 n z_0}}{i\beta_1 n} \frac{\lambda_n}{\beta_2 n} [y(\beta_1, \beta_2, \beta_3, z)]_{\lambda=\lambda_n} \quad (3.3-3)$$

where

$$y(\beta_1, \beta_2, \beta_3, z) = \frac{[x(\lambda) + n(\lambda)]\beta_1 \sin \beta_2 z - \delta_1 n(\lambda)\beta_2 \cos \beta_2 z}{- \frac{1}{\lambda_n} \frac{\partial}{\partial \lambda} [(\beta_2 C + i\beta_3 S)(\delta_3 \beta_2 C - \beta_3 S)(\beta_1 + U_2)(\beta_1 + X_2)]} ,$$

in which

$$x(\lambda) = 2(i - \delta_1)(\beta_2 C + i\beta_3 S)(\delta_3 \beta_2 C - \beta_3 S)(\beta_1 + X_2) ,$$

$$n(\lambda) = 2\beta_1 [(i - \delta_1)(\beta_3 S - \delta_3 \beta_2 C)(\beta_2 C + i\beta_3 S) - \delta_1(i + \delta_3)\beta_2^2]$$

#### 3.4 Application to the Water-Air-Ionosphere Problem.

We shall now consider the problem at hand. The question as to whether important modes of propagation are excited will be taken up first, since it has a bearing on the fields in all regions. It has already been asserted that only one important mode is expected to exist, due to the physical nature of the situation. Therefore we need to test the integrands of the expressions for  $\Pi_x$  and  $\Pi_z$  to determine whether the pole  $\lambda = \lambda_0$  occurs.

In the case of  $\Pi_x$ , this implies that we must look for the corresponding zero of  $(\beta_1 + U_2)$ . The small argument approximation for  $\tan \beta_2 h$  is made, and the resulting equation is

$$h(\beta_1 \beta_3 + \beta_2^2) \approx i(\beta_1 + \beta_3) .$$

When this is solved for  $\lambda$ , it is found that the value obtained is far in excess of that which could be justified in making the approximation to the tangent. The conclusion is that the integrand has no pole in the vicinity of  $k_2$ .

Such a result could have been anticipated on the grounds that we really cannot expect a horizontal component of the Hertz vector to excite a vertically polarized mode of propagation.  $\Pi_x$  can give rise only to horizontal components of electric field, and these are highly damped in the "waveguide" region because of the requirement that tangential E-fields be quite small on the interfaces.

For the case of the  $\Pi_z$  component, however, a factor  $(\beta_1 + X_2) = -i \sigma_1 (K_1 + Z_2)$  occurs in the denominator. Obviously the mode connected with the pole  $\lambda_0$  will be excited by this component.

The expressions for the mode contributions in regions 1 and 2 will be written later. The branch line integrals will now be investigated.

### 3.5 The Hertz Potential in Water

We take up the x-component first. Referring to (3.2-1) it is found that the part of the integrand in square brackets is

$$\frac{-4\beta_1\beta_2^2\beta_3}{\beta_2^2(\beta_1c+i\beta_2s)^2 - \beta_3^2(\beta_2c+i\beta_1s)^2}$$

It will again be necessary to resort to the asymptotic approximation to the Hankel function

$$H_0^2(\lambda_0) \approx \sqrt{\frac{2}{\pi\lambda_0}} e^{-i(\lambda_0 - \pi/4)} . \quad (3.5-1)$$

The integral becomes

$$I_{BL2} \approx 2M e^{i3\pi/4} \sqrt{\frac{2}{\pi\rho}} \int_{k_3 - i\infty}^{k_3} X(\lambda) e^{-i\lambda\rho} d\lambda , \quad (3.5-2)$$

where

$$X(\lambda) = \frac{\beta_2^2 \beta_3 \lambda^{1/2} e^{i\beta_1(z-z_0)}}{\beta_2^2 (\beta_1 C + i\beta_2 S)^2 - \beta_3^2 (\beta_2 C + i\beta_1 S)^2} .$$

When the substitution  $\lambda = a(1-i-it)$  is made, (with  $a = \text{Re } k_3$   
 $= \text{Im } k_3$ ) the factor  $\beta_3$  becomes

$$\sqrt{t} \sqrt{t+2(1+i)}$$

We can then write the integral in the form:

$$I_{BL2} \approx 2a^{5/2} M e^{i3\pi/4} \sqrt{\frac{2}{\pi\rho}} e^{-a\rho(1+i)} \int_0^\infty \sqrt{t} G(t) e^{-a\rho t} dt , \quad (3.5-3)$$

where

$$G(t) = \frac{\sqrt{t+2(1+i)}(1-i-it)^{1/2} \beta_2^2 e^{i\beta_1(z-z_0)}}{\beta_2^2 (\beta_1 C + i\beta_2 S)^2 - \beta_3^2 (\beta_2 C + i\beta_1 S)^2} ,$$

in which the  $\beta$ 's are understood to be expressed as functions of  $t$ . The first term in the Taylor series expansion of  $G(t)$  around the point  $t = 0$  is

$$G(0) = \frac{2 e^{ib_1(z-z_0)}}{(b_1 \bar{C} + ib_2 \bar{S})^2}$$

where  $b_n = \sqrt{k_n^2 + 2ia^2}$ ,  $n = 1, 2$ ;  $\bar{C} = \cos b_2 h$ ,  $\bar{S} = \sin b_2 h$ .

The corresponding first term of the integration is

$$I_{BL2}^{(1)} \approx -\sqrt{2} \operatorname{Me}^{i3\pi/4} a G(o) \frac{e^{-a\rho(1+i)}}{\rho^2} . \quad (3.5-4)$$

Succeeding terms will go as  $1/\rho^3$ ,  $1/\rho^4$ , etc. We conclude that the branch line integral associated with  $\beta_3$  vanishes at least as  $e^{a\rho}/\rho^2$  at large distances.

We will ignore the  $I_{BL3}$  contribution, since it will contain the term  $e^{-ik_1\rho}$ , and this must be negligible at appreciable distances.

Next we take up the branch line contribution for  $\Pi_z$  component of the Hertz potential. We refer to (3.2-6) and (3.1-30). Let  $\bar{\phi}_1$  be defined by

$$\phi_1(\beta_1, \beta_2, \beta_3) - \phi_1(\beta_1, \beta_2, -\beta_3) = \phi_1 - \bar{\phi}_1 , \quad (3.5-5)$$

where  $\phi_1$  is given by (3.1-30). Then

$$\phi_1 - \bar{\phi}_1 \approx 2 \beta_1 \delta_1 [(f-\bar{f}) - \delta_3 \beta_2^2 (g-\bar{g})] , \quad (3.5-6)$$

where we have put  $(\delta_1 - i) \approx \delta_1$  and  $(i + \delta_3) \approx \delta_3$ .

The term  $f-\bar{f}$  is defined by

$$f-\bar{f} = f(\beta_1, \beta_2, \beta_3) - f(\beta_1, \beta_2, -\beta_3) , \quad (3.5-7)$$

where

$$f(\beta_1, \beta_2, \beta_3) = \frac{1}{(\beta_1 + U_2)(\beta_1 + X_2)} ,$$

with  $U_2$  and  $X_2$  defined by (3.1-17) and (3.1-30). The term  $(g-\bar{g})$  is defined by

$$g-\bar{g} = g(\beta_1, \beta_2, \beta_3) - g(\beta_1, \beta_2, -\beta_3) \quad (3.5-8)$$

where

$$g(\beta_1, \beta_2, \beta_3) = \frac{1}{(\delta_3 \beta_2 C - \beta_3 S)(\beta_2 C + i \beta_3 S)(\beta_1 + U_2)(\beta_1 + X_2)}$$

It is found that to terms of order  $\beta_3$ :

$$[f-\bar{f}]_{\lambda \approx k_3} = \frac{-2\beta_3(1+T^2) [\beta_1(\delta_1+\delta_3) + \delta_1\beta_2T(\delta_3+1)]}{\delta_3(\beta_1+i\beta_2T)^2 (\beta_1+\delta_1\beta_2T)^2} \quad (3.5-9)$$

$$[g-\bar{g}]_{\lambda \approx k_3} = \frac{2\beta_2\beta_3G(\lambda)}{\delta_3^2\beta_2^4(\beta_1C+i\beta_2S)^2(\delta_1\beta_2S+\beta_1C)^2} \quad (3.5-10)$$

where

$$G(\lambda) = (\delta_1+\delta_3)\beta_1\beta_2C^2 + \delta_1\beta_2SC(i+\delta_3) + (i\delta_3-1)\beta_1^2SC + i(\delta_1\delta_3-1)\beta_1\beta_2S^2.$$

When approximations based on the relative values of the various quantities in the vicinity of  $\lambda = k_3$  are made, it is found further that

$$[f-\bar{f}]_{\lambda \approx k_3} \approx -\frac{2(\beta_1+\delta_3\beta_2T)\beta_3}{\delta_1\delta_3\beta_1^2\beta_2^2S^2} + O(\beta_3^3), \quad (3.5-11)$$

$$[g-\bar{g}]_{\lambda \approx k_3} \approx \frac{-2i\beta_3}{\delta_1\delta_3\beta_2^4\beta_1C^2} + O(\beta_3^3). \quad (3.5-12)$$

Terms of order  $\beta_3^3$  are dropped in our formulation because of the fact that the Taylor series expansion of part of the integrand is around the point  $\beta_3 = 0$ . Terms of order  $\beta_3$  contribute to the  $\rho^{-2}$  term; those of order  $\beta_3^3$  contribute to the  $\rho^{-3}$  terms, and so forth. We are confining our attention to the  $\rho^{-2}$  term.

Then,

$$[\theta_1-\bar{\theta}_1]_{\lambda \approx k_3} \approx \frac{4i\beta_3}{\beta_2^2C^2}. \quad (3.5-13)$$

Our approximations have reduced this quantity to quite a simple expression. Without such approximations, it would be extremely long and unwieldy.

The calculation of the branch line contribution to  $\Pi_z$  now proceeds:

$$K_{BL2} \approx \frac{M}{2} \cos \varnothing \int_C (\varnothing_1 - \bar{\varnothing}_1) e^{i\beta_1(z-z_0)} \frac{H_1^1(\lambda\rho)}{i\beta_1} \lambda^2 d\lambda, \quad (3.5-14)$$

where C denotes a small part of the branch cut in the vicinity of  $k_3$ . Then

$$K_{BL2} \approx \frac{2\sqrt{2} M \cos \varnothing}{k_1 \sqrt{\pi\rho}} e^{i\pi/4} a^{7/2} e^{-a\rho(1+i)} \int_0^\infty \sqrt{t} J(t) e^{-a\rho t} dt, \quad (3.5-15)$$

in which

$$J(t) = \frac{[t+2(1+i)]^{1/2} (1-i-it)^{3/2}}{\beta_2^2 c^2} e^{i\beta_1(z-z_0)}.$$

Then

$$J(0) = \frac{2(1-i)}{b_2^2 c^2} e^{ib_1(z-z_0)},$$

and the first term approximation to the integral is

$$K_{BL2} \approx M \cos \varnothing \frac{4a^2}{k_1 b_2^2 c^2} e^{ik_1(z-z_0)} \frac{e^{-a\rho(1+i)}}{\rho^2}. \quad (3.5-16)$$

As already mentioned, the  $\Pi_x$  component does not give rise to any freely propagating modes, but the  $\Pi_z$  component excites exactly the same mode as the vertical electric dipole. The denominator terms other than  $(\beta_1+U_2)$  and  $(\beta_1+X_2)$  merely cancel the common denominator of  $(\beta_1+U_2)$  and  $(\beta_1+X_2)$ . The determination of the mode solutions in the sea then resolves to finding the residue of the integrand of (3.1-25) at the pole  $\lambda=\lambda_0$ .

Referring to (3.2-7), the contribution from this pole is given by:

$$2\pi M \cos \theta \lambda_0 H_1^2(\lambda_0 \rho) e^{i\beta_1^0(z-z_0)} \frac{\Gamma(\lambda_0)}{(-1/\lambda_0)(\beta_1+x_2)'_{\lambda=\lambda_0}}, \quad (3.5-17)$$

where

$$\Gamma(\lambda) = \frac{(i-\delta_1)(\beta_3 s - \delta_3 \beta_2 c)(\beta_2 c + i\beta_3 s) - \delta_1(i+\delta_3)\beta_2^2}{(\delta_3 \beta_2 c - \beta_3 s)[\beta_1(\beta_2 c + i\beta_3 s) + \beta_2(\beta_3 c + i\beta_2 s)]}.$$

The factor  $-1/\lambda_0$  has been inserted in the denominator of (3.5-17) in order to cancel the common factor  $-\lambda_0$  which will result from the indicated differentiation and evaluation at  $\lambda_0$ . This completes our investigation of the Hertz potential in the water.

### 3.6 The Hertz Potential in Air.

We refer to equation (3.3-1). Under the conditions of this problem, we find that

$$V_2(\beta_1, \beta_2, \beta_3) - V_2(\beta_1, \beta_2, -\beta_3) = \frac{2\beta_2^2 \beta_3 (\beta_1 - \beta_2)}{\beta_2^2 (\beta_1 c + i\beta_2 s)^2 - \beta_3^2 (\beta_2 c + i\beta_1 s)^2}, \quad (3.6-1)$$

and that

$$W_2(\beta_1, \beta_2, \beta_3) - W_2(\beta_1, \beta_2, -\beta_3) = \frac{-2\beta_2^2 \beta_3 (\beta_1 + \beta_2)}{\beta_2^2 (\beta_1 c + i\beta_2 s)^2 - \beta_3^2 (\beta_2 c + i\beta_1 s)^2}. \quad (3.6-2)$$

The branch line integral is therefore (asymptotically)

$$I_{BL2} \approx -2 i M \sqrt{\frac{2}{\pi \rho}} e^{i\pi/4} \int_{k_3 - i\infty}^{k_3} Y(\lambda) e^{-i\lambda \rho} d\lambda, \quad (3.6-3)$$

where

$$Y(\lambda) = \frac{\lambda^{1/2} \beta_2^2 \beta_3 (\beta_1 \cos \beta_2 z + i \beta_2 \sin \beta_2 z) e^{-i \beta_1 z_0}}{\beta_2^2 (\beta_1 c + i \beta_2 s)^2 - \beta_3^2 (\beta_2 c + i \beta_1 s)^2}.$$

With the change of integration parameter as before, we have

$$I_{BL2} \approx 2a^2 M \sqrt{\frac{2}{\pi \rho}} e^{i\pi/4} e^{-a\rho(1+i)} \int_0^\infty \sqrt{t} s(t) e^{-a\rho t} dt, \quad (3.6-4)$$

where

$$s(t) = \frac{\sqrt{a}(1-i-it)^{1/2} e^{-i\beta_1 z_0} \beta_2^2 (\beta_1 \cos \beta_2 z + i \beta_2 \sin \beta_2 z) \sqrt{t+2(1+i)}}{\beta_2^2 (\beta_1 c + i \beta_2 s)^2 - \beta_3^2 (\beta_2 c + i \beta_1 s)^2}$$

Then

$$s(0) = \frac{2 \sqrt{a} e^{-i\beta_1 z_0} (\beta_1 \cos \beta_2 z + i \beta_2 \sin \beta_2 z)}{(\beta_1 c + i \beta_2 s)^2}$$

where the quantities are defined as before.

The first term in the approximation to the integral is therefore:

$$I_{BL2} \approx \sqrt{2} M e^{i\pi/4} a^{1/2} s(0) \frac{e^{-a\rho(1+i)}}{\rho^2} \quad (3.6-5)$$

This completes our investigation of the  $\Pi_x$  component.

We next evaluate the integrals for  $\Pi_z$ . We refer to (3.3-2) in connection with (3.1-31) and (3.1-32). Since we have already found  $\varnothing(\beta_1, \beta_2, \beta_3) - \varnothing_1(\beta_1, \beta_2, -\beta_3)$ , the most involved part of the algebra is done. Making approximations as before based on the physical value of the parameters in the region of greatest contribution to the integral, we find that the integrand of (3.3-2) is given, to terms of order  $\beta_3$ , by

$$\frac{4i\delta_1\beta_3}{\beta_1^2\beta_2^2c^2} (\beta_2 \cos \beta_2 z + i\beta_1 \sin \beta_2 z) \lambda^2 H_1^2(\lambda\rho) e^{-i\beta_1 z_0} \quad (3.6-6)$$

with the integration to be taken from  $k_3 - i\infty$  to  $k_3$ . The asymptotic form of the Hankel function is

$$H_1^2(\lambda\rho) \approx \sqrt{\frac{2}{\pi\lambda\rho}} e^{-i(\lambda\rho - 3\pi/4)} \quad (3.6-7)$$

The change of integration parameter is again  $\lambda = a(1-i-it)$ , and the integral becomes approximately

$$-2\sqrt{\frac{2}{\pi}} \delta_1 M \cos \varnothing a^{7/2} \frac{e^{-a\rho(1+i)}}{\sqrt{\rho}} \int_0^\infty \sqrt{t} P(t,z) e^{-a\rho} dt, \quad (3.6-8)$$

where

$$P(t,z) = \frac{(\beta_2 \cos \beta_2 z + i\beta_1 \sin \beta_2 z)(1-i-it)^{3/2} [t+2(1+i)]^{1/2} e^{-i\beta_1 z} o_{+0}(t)}{\beta_1^2 \beta_2^2 \bar{c}^2}$$

Dropping the  $O(t)$  term, which would give rise to terms in  $\rho^{-3}$

$$F_1(0,z) = 2 \frac{b_2 \cos b_2 z + i b_1 \sin b_2 z}{b_1^2 b_2^2 \bar{c}^2} (1-i) e^{i b_1 z} o.$$

The first term asymptotic approximation to the integral is therefore

$$-4M \cos \varnothing \delta_1 a^2 e^{-i\pi/4} e^{-i b_1 z} o \frac{b_2 \cos b_2 z + i b_1 \sin b_2 z}{b_1^2 b_2^2 \bar{c}^2} \frac{e^{-a\rho(1+i)}}{\rho^2} \quad (3.6-9)$$

For the residue at the pole we consider equation (3.3-3).

Noting that  $X_2 = -\beta_1$  at the pole  $\lambda = \lambda_o$ , we obtain for the residue at this pole:

$$2 i\pi M \cos \varnothing \lambda_o H_1^2(\lambda_o \rho) \frac{e^{-i\beta_1^o z} o}{\beta_2^o} \frac{r(\lambda_o, z)}{(\beta_1 + X_2)'_{\lambda=\lambda_o}}, \quad (3.6-10)$$

where  $r(\lambda, z) =$

$$\frac{[(i-\delta_1)(\beta_3 S - \delta_3 \beta_2 C)(\beta_2 C + i\beta_3 S) - \delta_1(i+\delta_3)\beta_2^2](\beta_1 \sin \beta_2 z - \delta_1 \beta_2 \cos \beta_2 z)}{(-1/\lambda_0)(\beta_2 C + i\beta_3 S)(\delta_3 \beta_2 C - \beta_3 S)(\beta_1 + U_2)}$$

The fields in air due to the submerged horizontal electric dipole are now specified by (3.6-5), (3.6-9) and (3.6-11).

#### 4.0 DERIVATION OF FIELD COMPONENTS

##### 4.1 The Vertical Electric Dipole.

From (2.5-23) the mode solution in water is

$$- \frac{2i\pi M}{\sigma_1} \lambda_o^2 H_o^2(\lambda_o) \frac{e^{i\beta_1(z-z_o)}}{(K_1+Z_2)'_{\lambda=\lambda_o}}$$

The field components are derived as indicated in (2.1-22) and are found to be as follows. The superscript (v,m) will be used to denote "vertical dipole, mode solution", and (v,b) for "vertical dipole, branch line solution". In most cases the approximation  $\beta_1 \approx k_1$  will be made except when the term  $k_1^2 - \beta_1^2$  arises. In this case the exact difference,  $\lambda_o^2$ , must of course be used.

$$\begin{aligned} E_{1\phi}^{(v,m)} &= \frac{2\pi M}{\sigma_1} \lambda_o^2 k_1 H_1^2(\lambda_o) \frac{e^{ik_1(z-z_o)}}{(K_1+Z_2)'_{\lambda=\lambda_o}} , \\ E_{1z}^{(v,m)} &= - \frac{2i\pi M}{\sigma_1} \lambda_o^3 H_o^2(\lambda_o) \frac{e^{ik_1(z-z_o)}}{(K_1+Z_2)'_{\lambda=\lambda_o}} , \\ H_{1\phi}^{(v,m)} &= 2i\pi M \lambda_o^2 H_1^2(\lambda_o) \frac{e^{ik_1(z-z_o)}}{(K_1+Z_2)'_{\lambda=\lambda_o}} . \end{aligned} \quad (4.1-1)$$

The branch line contributions are obtained by differentiation of the expression (2.5-15). The result is that the fields are given by appropriate factors multiplying  $\bar{I}_{BL2}$ . We find:

$$\begin{aligned} E_{1\phi}^{(v,b)} &\approx a k_1 (1-i) \bar{I}_{BL2} , \\ E_{1z}^{(v,b)} &\approx (k_1^2 - b_1^2) \bar{I}_{BL2} , \\ H_{1\phi}^{(v,b)} &\approx - \frac{k_1^2}{\omega \mu_o} a (1-i) \bar{I}_{BL2} . \end{aligned} \quad (4.1-2)$$

Therefore

$$\begin{aligned}
 E_{1\rho}(v,b) &\approx + \frac{4Mk_1 a^2}{\delta_1 \delta_3 b_2^2 \bar{s}^2} e^{ik_1(z-z_0)} \frac{e^{-a\rho(1+i)}}{\rho^2} , \\
 E_{1z}(v,b) &\approx \frac{4\sqrt{2} M a^3 e^{-i\pi/4}}{\delta_1 \delta_3 b_2^2 \bar{s}^2} e^{ik_1(z-z_0)} \frac{e^{-a\rho(1+i)}}{\rho^2} , \\
 H_{1\theta}(v,b) &\approx - \frac{2M\omega^3 \mu_0 \epsilon_0^2}{b_2^2 \bar{s}^2} e^{ik_1(z-z_0)} \frac{e^{-a\rho(1+i)}}{\rho^2} .
 \end{aligned} \tag{4.1-3}$$

The fields expressed by equations (4.1-3) must be added to those of equations (4.1-1).

For the fields in air we refer to equations (2.6-10) and (2.6-9) for the mode solution and branch line solution, respectively. The mode solution fields become:

$$\begin{aligned}
 E_{2\rho}(v,m) &= - \frac{2M\pi}{\sigma_1} \lambda_0 H_1^2(\lambda_0 \rho) e^{-i\beta_1 z_0} \frac{q(\lambda_0, z)}{(-1/\lambda_0)(K_1 + Z_2)'_{\lambda=\lambda_0}} , \\
 E_{2z}(v,m) &= - \frac{2M\pi}{\sigma_1} \lambda_0^2 H_0^2(\lambda_0 \rho) e^{-i\beta_1 z_0} \frac{p(\lambda_0, z)}{(-1/\lambda_0)(K_1 + Z_2)'_{\lambda=\lambda_0}} , \\
 H_{2\theta}(v,m) &= - \frac{2iM\pi}{\delta_1} \lambda_0 H_1^2(\lambda_0 \rho) e^{-i\beta_1 z_0} \frac{p(\lambda_0, z)}{(-1/\lambda_0)(K_1 + Z_2)'_{\lambda=\lambda_0}} ,
 \end{aligned} \tag{4.1-4}$$

where

$$q(\lambda, z) = \delta_1 \beta_2 \sin \beta_2 z - \beta_1 \cos \beta_2 z$$

$$p(\lambda, z) = \delta_1 \cos \beta_2 z + \beta_1 / \beta_2 \sin \beta_2 z .$$

The branch line contribution to the fields in air can be obtained from the appropriate differentiation of (2.6-9). This

results in some simple modifications, thus:

$$\begin{aligned}
 E_{2\rho}^{(v,b)} &\approx \frac{4Ma^2 e^{+i3\pi/4} e^{-ik_1 z_0} e^{-a\rho(1+i)}}{\delta_1 \delta_3 b_2^2 S^2} \frac{e^{-a\rho(1+i)}}{\rho^2} (b_1 \cos b_2 z + \delta_1 b_2 \sin b_2 z), \\
 E_{2z}^{(v,b)} &\approx - \frac{4\sqrt{2} Ma^3 e^{-ik_1 z_0} e^{-a\rho(1+i)}}{\delta_1 \delta_3 b_2^2 S^2} \frac{e^{-a\rho(1+i)}}{\rho^2} (b_1 \sin b_2 z - \delta_1 b_2 \cos b_2 z), \\
 H_{2\phi}^{(v,b)} &\approx \frac{4M k_2^2 a^2 e^{-i\pi/4}}{\omega \mu_0 \delta_1 \delta_3 b_2^2 S^2} e^{-ik_1 z_0} \frac{e^{-a\rho(1+i)}}{\rho^2} (b_1 \sin b_2 z - \delta_1 b_2 \cos b_2 z).
 \end{aligned}
 \tag{4.1-5}$$

#### 4.2 The Horizontal Electric Dipole.

Equation (3.5-17) gives the mode solution in water for the horizontal dipole. Equations (2.1-22) prescribe the determination of the fields, which are found to be:

$$\begin{aligned}
 E_{1\rho}^{(h,m)} &= Mik_1 \pi \cos \phi \lambda_0^2 [H_0^2(\lambda_0 \rho) - H_2^2(\lambda_0 \rho)] e^{ik_1(z-z_0)} \frac{\Gamma(\lambda_0)}{(-1/\lambda_0)(\beta_1 + X_2)}, \\
 E_{1z}^{(h,m)} &= 2M\pi \cos \phi H_1^2(\lambda_0 \rho) \lambda_0^3 e^{ik_1(z-z_0)} \frac{\Gamma(\lambda_0)}{(-1/\lambda_0)(\beta_1 + X_2)}, \\
 H_{1\phi}^{(h,m)} &= -\pi M \sigma_1 \cos \phi \lambda_0^2 [H_0^2(\lambda_0 \rho) - H_2^2(\lambda_0 \rho)] e^{ik_1(z-z_0)} \frac{\Gamma(\lambda_0)}{(-1/\lambda_0)(\beta_1 + X_2)},
 \end{aligned}
 \tag{4.1-6}$$

The superscript (h,m) denotes "horizontal dipole, mode solution," while (h,b) denotes "horizontal dipole, branch line solution."

Contributions from integration along the branch cuts are obtained from the following equations, where  $\Pi_{x1}^b$  is (3.5-4) and  $\Pi_{z1}^b$  is (3.5-16):

$$\begin{aligned}
 E_{1\rho}^{(h,b)} &= \cos \varnothing \left[ (k_1^2 + \frac{\partial^2}{\partial \rho^2}) \Pi_{x1}^b \right] + \frac{\partial^2 \Pi_{z1}^b}{\partial \rho \partial z} , \\
 E_{1\varnothing}^{(h,b)} &= - \sin \varnothing (k_1^2 + \frac{2}{\rho} \frac{\partial}{\partial \rho}) \Pi_{x1}^b + \frac{1}{\rho} \frac{\partial^2 \Pi_{z1}^b}{\partial z \partial \varnothing} , \\
 E_{1z}^{(h,b)} &= k_1^2 \Pi_{z1}^b + \cos \varnothing \frac{\partial^2 \Pi_{x1}^b}{\partial \rho \partial z} + \frac{\partial^2 \Pi_{z1}^b}{\partial z^2} , \quad (4.1-7) \\
 H_{1\rho}^{(h,b)} &= \frac{k_1^2}{i\omega\mu_0} \left( \frac{1}{\rho} \frac{\partial \Pi_{z1}^b}{\partial \varnothing} + \sin \varnothing \frac{\partial \Pi_{x1}^b}{\partial z} \right) , \\
 H_{1\varnothing}^{(h,b)} &= \cos \varnothing \frac{\partial \Pi_{x1}^b}{\partial z} - \frac{\partial \Pi_{z1}^b}{\partial \rho} , \\
 H_{1z}^{(h,b)} &= - \sin \varnothing \frac{\partial \Pi_{x1}^b}{\partial \rho} .
 \end{aligned}$$

It is readily found that to terms of order  $\rho^{-2}$ :

$$\begin{aligned}
 E_{1\rho}^{(h,b)} &= \cos \varnothing (k_1^2 + 2ia^2) \Pi_{x1}^b + (1-i)k_1 a \Pi_{z1}^b , \\
 E_{1\varnothing}^{(h,b)} &= - \sin \varnothing k_1^2 \Pi_{x1}^b , \\
 E_{1z}^{(h,b)} &= \cos \varnothing k_1 a (1-i) \Pi_{x1}^b , \quad (4.1-8) \\
 H_{1\rho}^{(h,b)} &= \frac{k_1^3}{\omega\mu_0} \sin \varnothing \Pi_{x1}^b , \\
 H_{1\varnothing}^{(h,b)} &= ik_1 \cos \varnothing \Pi_{x1}^b + (1+i)a \Pi_{z1}^b , \\
 H_{1z}^{(h,b)} &= \sin \varnothing (1+i) a \Pi_{x1}^b .
 \end{aligned}$$

$\Pi_{x1}^b$  and  $\Pi_{z1}^b$  are relisted here for convenience:

$$\Pi_{x1}^b \approx -2\sqrt{2} a M e^{i3\pi/4} \frac{e^{ik_1(z-z_0)}}{(b_1\bar{C}+ib_2\bar{S})^2} \frac{e^{-a\rho(1+i)}}{\rho^2} , \quad (4.1-9)$$

$$\Pi_{z1}^b \approx \frac{4a^2 M \cos \varnothing}{k_1 b_2^2 \bar{C}^2} e^{ik_1(z-z_0)} \frac{e^{-a\rho(1+i)}}{\rho^2} \quad (4.1-10)$$

For the fields in air we deal with (3.6-10) for the mode solution, which gives:

$$E_{2\rho}^{(h,m)} = i\pi M \cos \varnothing \lambda_o^2 [H_o^2(\lambda_{op}) - H_2^2(\lambda_{op})] e^{-ik_1 z_o} \frac{s(\lambda_o, z)}{(\beta_1 + X_2)^{1/2}} \quad , \quad \lambda = \lambda_o$$

$$E_{2z}^{(h,m)} = 2i\pi M \cos \varnothing \lambda_o^3 H_1^2(\lambda_{op}) \frac{e^{-ik_1 z_o}}{\beta_2} \frac{r(\lambda_o, z)}{(\beta_1 + X_2)^{1/2}} \quad , \quad \lambda = \lambda_o$$

$$H_{2\varnothing}^{(h,m)} = -\omega \epsilon_o \pi M \cos \varnothing \lambda_o^2 [H_o^2(\lambda_{op}) - H_2^2(\lambda_{op})] \frac{e^{-ik_1 z_o}}{\beta_2} \frac{r(\lambda_o, z)}{(\beta_1 + X_2)^{1/2}} \quad , \quad \lambda = \lambda_o \quad (4.1-11)$$

in which

$$s(\lambda, z) = \frac{[(i - \delta_1)(\beta_3 S - \delta_3 \beta_2 C)(\beta_2 C + i\beta_3 S) - \delta_1(i + \delta_3)\beta_2^2](\beta_1 \cos \beta_2 z + \delta_1 \beta_2 \sin \beta_2 z)}{(-1/\lambda_o)(\beta_2 C + i\beta_3 S)(\delta_3 \beta_2 C - \beta_3 S)(\beta_1 + U_2)}$$

and

$$r(\lambda, z) = \frac{[(i - \delta_1)(\beta_3 S - \delta_3 \beta_2 C)(\beta_2 C + i\beta_3 S) - \delta_1(i + \delta_3)\beta_2^2](\beta_1 \sin \beta_2 z - \delta_1 \beta_2 \cos \beta_2 z)}{(-1/\lambda_o)(\beta_2 C + i\beta_3 S)(\delta_3 \beta_2 C - \beta_3 S)(\beta_1 + U_2)}$$

For the branch line solutions in air we refer to (3.6-5) and (3.6-9), which are rewritten here for convenience:

$$\Pi_{x2}^b = A_1 (b_1 \cos b_2 z + i b_2 \sin b_2 z) \frac{e^{-a\rho(1+i)}}{\rho^2} \quad , \quad (4.1-12)$$

$$\Pi_{z2}^b = A_2 \cos \varnothing (b_2 \cos b_2 z + i b_1 \sin b_2 z) \frac{e^{-a\rho(1+i)}}{\rho^2} \quad , \quad (4.1-13)$$

where

$$A_1 = \frac{2\sqrt{2} M e^{i\pi/4} a e^{-ib_1 z_o}}{(b_1 \bar{C} + i b_2 \bar{S})^2}$$

$$A_2 = \frac{4M \delta_1 a^2 e^{-i\pi/4} e^{-ib_1 z_o}}{b_1^2 b_2^2 \bar{C}^2} \quad .$$

The derivation of field components proceeds similarly to (4.1-7), giving to terms of order  $\rho^{-2}$ :

$$E_{2\rho}(h, \rho) = \cos \varnothing \frac{e^{-a\rho}(1+i)}{\rho^2} E_1(b_1 \cos b_2 z + ib_2 \sin b_2 z) \\ \text{(where } E_1 = (K_2^2 + 2ia^2) A_1 + (1+i) a b_2 A_2 \text{) ,}$$

$$E_{2\varnothing}(h, b) = - \sin \varnothing \frac{e^{-a\rho}(1+i)}{\rho^2} k_2 A_1 (b_1 \cos b_2 z + ib_2 \sin b_2 z) ,$$

$$E_{2z}(h, b) = \cos \varnothing \frac{e^{-a\rho}(1+i)}{\rho^2} B_2 (b_2 \cos b_2 z + ib_1 \sin b_2 z) , \quad (4.1-14)$$

$$\text{(where } B_2 = k_2^2 A_2 + (1-i) a b_2 A_1 \text{) ,}$$

$$H_{2\rho}(h, \rho) = - \sin \varnothing \frac{e^{-a\rho}(1+i)}{\rho^2} \frac{k_2^2 b_2 A_1}{i\omega\mu_0} (b_1 \sin b_2 z - ib_2 \cos b_2 z) ,$$

$$H_{2\varnothing}(h, \rho) = \cos \varnothing \frac{e^{-a\rho}(1+i)}{\rho^2} B_3 (b_2 \cos b_2 z + ib_1 \sin b_2 z) ,$$

$$\text{(where } B_3 = a(1+i)A_2 + ib_2 A_1 \text{) ,}$$

$$H_{2z}(h, b) = \sin \varnothing \frac{e^{-a\rho}(1+i)}{\rho^2} a(1+i) A_1 (b_1 \cos b_2 z + ib_2 \sin b_2 z) .$$

In all cases, as previously noted,  $b_1$  can be replaced by  $k_1$  with negligible error.

## 5.0 NUMERICAL COMPARISONS

Although the field equations thus far obtained do not appear particularly formidable, some of them involve rather long and tedious calculations in obtaining numerical results. Therefore, a set of simplified formulae were derived from the field equations for the purpose of easily obtaining numerical results, and are presented in Appendix B. The results are presented in the following tables. For detailed comparisons between the different antennas at different frequencies, it would be advisable to do a sufficiently complete series of computations so that curves could be drawn giving field strengths versus frequency.

The computations have been made for fields in water at frequencies 1, 10, 100, and 1000 cps. In dealing with the mode solutions it is necessary to pick a distance at which to make comparisons, and this has been chosen as 1000 km. Unfortunately, the small-argument approximations (first term of the power series) to the Hankel functions will be fairly good only for 1 cps, and the asymptotic approximation only for 1000 cps. Here we have extended the small argument approximation upward to 10 cps and the asymptotic approximation downward to 100 cps, but obviously the results cannot be expected to be very good for these two frequencies. They will not be too bad when only the  $H_0^2$  function is involved, but will be somewhat worse for the  $H_1^2$  and  $H_2^2$  functions.

We will list here numerical values for  $\lambda_0$  at the frequencies mentioned, since these will be used considerably

in succeeding calculations. In all cases the pertinent constants are as given by (2.5-20).

<u>f, cps</u>	<u>Mod <math>\lambda_o</math></u>	<u>Arg <math>\lambda_o</math></u>
1	$3.02 \times 10^{-8}$	$-12.55^\circ$
10	$2.4 \times 10^{-7}$	$-6.13^\circ$
100	$2.19 \times 10^{-6}$	$-2.12^\circ$
1000	$2.13 \times 10^{-5}$	$-0.78^\circ$

We note that when  $\rho = 10^6$  meters,

$$\lambda_{o\rho} = \left\{ \begin{array}{ll} 3.02 \times 10^{-2} & \exp(-i12.55^\circ) \\ 0.24 & \exp(-i6.13^\circ) \\ 2.185 & \exp(-i2.12^\circ) \\ 21.25 & \exp(-i0.78^\circ) \end{array} \right\}$$

### 5.1 Vertical dipole, Mode Solutions.

With the understanding that in the following groups of four, the values for  $f = 1, 10, 100$ , and  $1000$  cps are from top to bottom and that the first term of the power series for the Hankel function is used at  $f = 1, 10$  cps, and the first term of the asymptotic expansion is used at  $f = 100, 1000$  cps, the vertical dipole, mode solution results are for  $z = 0$  and  $\rho = 1000$  km:

$$|E_1^{(v,m)}| = \left\{ \begin{array}{l} 1.76 \times 10^{-24} \\ 5.48 \times 10^{-23} \\ 3.22 \times 10^{-21} \\ 2.98 \times 10^{-19} \end{array} \right\} M |e^{-ik_1 z_o}|$$

$$|E_{1z}^{(v,m)}| = \begin{Bmatrix} 1.11 \times 10^{-30} \\ 3.8 \times 10^{-28} \\ 1.25 \times 10^{-25} \\ 3.78 \times 10^{-23} \end{Bmatrix} \quad M |e^{-ik_1 z_0}|$$

$$|H_{1\theta}^{(v,m)}| = \begin{Bmatrix} 1.24 \times 10^{-21} \\ 1.23 \times 10^{-20} \\ 2.3 \times 10^{-19} \\ 6.7 \times 10^{-18} \end{Bmatrix} \quad M |e^{-ik_1 z_0}|$$

## 5.2 Horizontal dipole, Mode solutions.

For the horizontal electric dipole mode solutions we find for the magnitudes at  $z = 0$ ,  $\rho = 1000$  km, and the previously mentioned approximations to the Hankel functions:

$$|E_1^{(h,m)}| = \begin{Bmatrix} 1.13 \times 10^{-17} \\ 1.75 \times 10^{-17} \\ 8.92 \times 10^{-17} \\ 2.71 \times 10^{-15} \end{Bmatrix} \quad M \cos \theta |e^{-k_1 z_0}|$$

$$|E_{1z}^{(h,m)}| = \begin{Bmatrix} 1.84 \times 10^{-24} \\ 5.45 \times 10^{-23} \\ 3.48 \times 10^{-21} \\ 3.05 \times 10^{-19} \end{Bmatrix} \quad M \cos \theta |e^{-ik_1 z_0}|$$

$$|H_{1\theta}^{(h,m)}| = \begin{Bmatrix} 8.05 \times 10^{-15} \\ 3.96 \times 10^{-15} \\ 6.35 \times 10^{-15} \\ 6.12 \times 10^{-14} \end{Bmatrix} \quad M \cos \theta |e^{-ik_1 z_0}|$$

It is seen that for  $z = 0$ , the ratio of mode fields from the horizontal dipole to those from the vertical dipole ranges from the order of  $10^7$  to  $10^3$ , depending on frequency and component. Evidently the horizontal dipole is more efficient in exciting the mode concerned in this case.

### 5.3 Vertical dipole, branch line solutions.

In calculating the branch line integrals, the following will be useful:

$$\bar{S} = \sin b_2 h = \begin{cases} 0.801 \exp(-i39^\circ) \\ 3.07 \exp(-i12.5^\circ) \\ 1.42 \times 10^2 \exp(i233.85^\circ) \\ 5.94 \times 10^7 \exp(i214.4^\circ) \end{cases}$$

$$\bar{C} = \cos b_2 h = \begin{cases} 1.035 \exp(i17.9^\circ) \\ 2.91 \exp(i103.6^\circ) \\ 1.42 \times 10^2 \exp(-i36.1^\circ) \\ 5.94 \times 10^7 \exp(-i55.28^\circ) \end{cases}$$

It is found that:

$$E_{1\rho}(v,b) = \left\{ \begin{array}{l} 2.61 \times 10^{-18} \exp(-i57^\circ) \\ 5.6 \times 10^{-18} \exp(-i110.7^\circ) \\ 8.33 \times 10^{-20} \exp(-i242.7^\circ) \\ 1.5 \times 10^{-29} \exp(-i203.8^\circ) \end{array} \right\} M e^{ik_1(z-z_0)} \frac{e^{-a\rho(1+i)}}{\rho^2}$$

$$E_{1z}(v,b) = \left\{ \begin{array}{l} 4.11 \times 10^{-21} \exp(-i57^\circ) \\ 8.86 \times 10^{-21} \exp(-i110.7^\circ) \\ 1.31 \times 10^{-22} \exp(-i242.7^\circ) \\ 3.79 \times 10^{-32} \exp(-i203.8^\circ) \end{array} \right\} M e^{ik_1(z-z_0)} \frac{e^{-a\rho(1+i)}}{\rho^2}$$

$$H_{10}(v,b) = \left\{ \begin{array}{l} 1.85 \times 10^{-15} \exp(-i102^\circ) \\ 3.99 \times 10^{-15} \exp(-i155.7^\circ) \\ 5.93 \times 10^{-17} \exp(-i72.3^\circ) \\ 1.07 \times 10^{-26} \exp(-i111.2^\circ) \end{array} \right\} Me^{ik_1(z-z_0)} \frac{e^{-a\rho(1+i)}}{\rho^2}$$

At  $\rho = 1000$  km, the magnitude of the factor  $\frac{e^{-a\rho}}{\rho^2}$  is, for the four frequencies:

$$1.88 \times 10^{-15}$$

$$2.33 \times 10^{-21}$$

$$5.21 \times 10^{-40}$$

- - - - -

The field components at  $\rho = 1000$  km become, listing only magnitudes:

$$|E_{1\rho}(v,b)| = \left\{ \begin{array}{l} 4.9 \times 10^{-33} \\ 1.37 \times 10^{-35} \\ 4.33 \times 10^{-59} \\ - - - - - \end{array} \right\} |Me^{ik_1(z-z_0)}|$$

$$|E_{1z}(v,b)| = \left\{ \begin{array}{l} 7.72 \times 10^{-36} \\ 2.06 \times 10^{-41} \\ 6.85 \times 10^{-62} \\ - - - - - \end{array} \right\} |Me^{ik_1(z-z_0)}|$$

$$|H_{1\theta}(v,b)| = \left\{ \begin{array}{l} 3.48 \times 10^{-30} \\ 9.3 \times 10^{-36} \\ 3.09 \times 10^{-56} \\ - - - - - \end{array} \right\} |Me^{ik_1(z-z_0)}|$$

These fields are negligible compared with the mode solution values.

#### 5.4 Horizontal dipole, Branch line solutions.

Branch line contributions for the horizontal electric dipole are calculated from equations (4.1-8), and are determined to be:

$$E_{1\rho}(h,b) = \left\{ \begin{array}{l} 2.41 \times 10^{-5} \exp(-i35.8^\circ) \\ 1.32 \times 10^{-5} \exp(-i90^\circ) \\ 1.31 \times 10^{-8} \exp(i72.2^\circ) \\ 2.36 \times 10^{-19} \exp(i111^\circ) \end{array} \right\} M \cos \theta e^{ik_1(z-z_o)} \frac{e^{-a\rho(1+i)}}{\rho^2}$$

$$E_{1\theta}(h,b) = \left\{ \begin{array}{l} 1.75 \times 10^{-5} \exp(i99.2^\circ) \\ 6.97 \times 10^{-6} \exp(-i72.2^\circ) \\ 9.25 \times 10^{-9} \exp(i207.2^\circ) \\ 1.67 \times 10^{-19} \exp(i246^\circ) \end{array} \right\} M \sin \theta e^{ik_1(z-z_o)} \frac{e^{-a\rho(1+i)}}{\rho^2}$$

$$E_{1z}(h,b) = \left\{ \begin{array}{l} 2.87 \times 10^{-8} \exp(-i80.8^\circ) \\ 1.14 \times 10^{-8} \exp(-i252.2^\circ) \\ 1.53 \times 10^{-11} \exp(i27.2^\circ) \\ 2.78 \times 10^{-22} \exp(i66^\circ) \end{array} \right\} M \cos \theta e^{ik_1(z-z_o)} \frac{e^{-a\rho(1+i)}}{\rho^2}$$

$$H_{1\rho}(h,b) = \left\{ \begin{array}{l} 1.25 \times 10^{-2} \exp(-i125.8^\circ) \\ 1.58 \times 10^{-3} \exp(i62.8^\circ) \\ 6.65 \times 10^{-7} \exp(i17.8^\circ) \\ 3.81 \times 10^{-18} \exp(i21^\circ) \end{array} \right\} M \sin \theta e^{ik_1(z-z_o)} \frac{e^{-a\rho(1+i)}}{\rho^2}$$

$$H_{1\theta}(h,b) = \left\{ \begin{array}{l} 4.39 \times 10^{-3} \exp(i99.2^\circ) \\ 5.57 \times 10^{-4} \exp(-i72.2^\circ) \\ 1.7 \times 10^{-7} \exp(i196.9^\circ) \\ 1.26 \times 10^{-18} \exp(-i121.5^\circ) \end{array} \right\} M \cos \theta e^{ik_1(z-z_o)} \frac{e^{-a\rho(1+i)}}{\rho^2}$$

$$H_{1z}(h,b) = \left\{ \begin{array}{l} 3.46 \times 10^{-6} \exp(i99.2^\circ) \\ 4.37 \times 10^{-7} \exp(-i72.2^\circ) \\ 1.84 \times 10^{-10} \exp(i207.2^\circ) \\ 1.05 \times 10^{-21} \exp(i246^\circ) \end{array} \right\} M \sin \theta e^{ik_1(z-z_0)} \frac{e^{-a\rho(1+i)}}{\rho^2}$$

The branch line solutions to the horizontal electric dipole are evidently much greater in magnitude than those for the vertical electric dipole, at low frequencies, and can therefore be expected to have considerable importance at close ranges. Unfortunately the approximations made in evaluating the branch line integrals are not very good at close ranges. For validity within an order of magnitude, we might put bounds as follows on the distances:

$$\begin{aligned} f = 1 & \quad \rho \gtrsim 10^6 \text{ meters ,} \\ f = 10 & \quad \rho \gtrsim 3 \times 10^5 \text{ meters ,} \\ f = 100 & \quad \rho \gtrsim 10^5 \text{ meters ,} \\ f = 1000 & \quad \rho \gtrsim 3 \times 10^4 \text{ meters .} \end{aligned}$$

At  $\rho = 1000$  km the field components due to the branch line solutions to the horizontal dipole are, in magnitude:

$$E_{1\rho}(h,b) = \left\{ \begin{array}{l} 4.53 \times 10^{-20} \\ 3.08 \times 10^{-26} \\ 6.83 \times 10^{-48} \\ \text{---} \end{array} \right\} M \cos \theta e^{ik_1(z-z_0)} ,$$

$$E_{1\theta}(h,b) = \left\{ \begin{array}{l} 3.29 \times 10^{-20} \\ 1.62 \times 10^{-26} \\ 4.81 \times 10^{-48} \\ \text{---} \end{array} \right\} M \sin \theta e^{ik_1(z-z_0)} ,$$

$$E_{1z}(h,b) = \left\{ \begin{array}{l} 5.4 \times 10^{-26} \\ 2.66 \times 10^{-29} \\ 7.97 \times 10^{-51} \\ \text{---} \end{array} \right\} M \cos \theta e^{ik_1(z-z_0)} ,$$

$$H_{1\rho}(h,b) = \left\{ \begin{array}{l} 2.35 \times 10^{-17} \\ 3.68 \times 10^{-24} \\ 3.47 \times 10^{-46} \\ \text{---} \end{array} \right\} M \sin \theta e^{ik_1(z-z_0)} ,$$

$$H_{1\theta}(h,b) = \left\{ \begin{array}{l} 8.25 \times 10^{-18} \\ 1.3 \times 10^{-24} \\ 8.86 \times 10^{-46} \\ \text{---} \end{array} \right\} M \cos \theta e^{ik_1(z-z_0)} ,$$

$$H_{1z}(h,b) = \left\{ \begin{array}{l} 6.51 \times 10^{-21} \\ 1.02 \times 10^{-27} \\ 9.59 \times 10^{-50} \\ \text{---} \end{array} \right\} M \sin \theta e^{ik_1(z-z_0)} .$$

## 6.0 ANTENNAS

### 6.1 General Discussion.

The requirements for a submerged ELF electric antenna can be summed up qualitatively in a rather brief manner. The fundamental necessity is that of obtaining the greatest possible electric moment. This moment is proportional to the product of current and antenna length; hence the goal is to use the greatest current and greatest length possible. Because, however, the surrounding conducting medium imposes a penalty in the form of high losses in any region where field intensity is high, the maximization of current must be achieved by some method which maintains field intensity as low as possible everywhere in the water.

This problem can be viewed from another standpoint, namely that of impedance matching. The intrinsic wave impedance of sea water is very low, which is essentially the same as saying that the ratio of electric to magnetic field is low. For good impedance matching from antenna to water, it follows that the electric fields should be kept low while at the same time generating currents of the maximum possible magnitude.

If the additional requirement is made that the antenna be readily transportable, other restrictions are immediately evident. Extremely large spherical, spheroidal, or other shapes with cross sections which preclude good hydrodynamic properties are excluded.

In view of the above considerations it would seem that the only really feasible submarine antenna for ELF is one involving a trailing wire of some sort. Under various circumstances, objections can be raised to such an antenna; nevertheless, there seems to be no other type for which much more serious disadvantages do not appear. In the following, we will confine our attentions to this kind of antenna and more specifically to the "coaxial" antenna about to be described.

The problem has already been considered in considerable detail by Moore,<sup>4</sup> who introduced the coaxial antenna. This is an insulated length of wire, the input end of which is fastened to the submarine. The trailing end may be either insulated or exposed to the sea water. In the first case the system resembles a coaxial transmission line with the load end open circuited. In the second case it resembles a short-circuited coaxial transmission line. To obtain an effective short circuit in the latter case, the amount of exposed wire necessary at the trailing end, as shown by Moore, need only be around one-eighth wavelength (in sea water).

For any reasonable overall length, considering the frequency range with which we are dealing, the short circuited antenna will carry essentially uniform current throughout its length. This can be considered due to the high velocity of propagation of this "transmission line". Such an antenna may, however, be of appreciable length compared with a wavelength in sea water. At 1000 cps, for example, a half wavelength in sea water is only about 17 meters. At 10 cycles per second it is

ten times as great, or 170 meters. Although this latter figure might appear to be a rather cumbersome length for a trailing wire, it is perhaps not beyond the realm of possibility.

The distribution of current in the open circuited coaxial antenna can be approximated as varying linearly from its maximum at the submarine or input end to zero at the trailing end. The effective current in this case is therefore half of the input current. Since the reactance of the open circuited antenna is quite high and the difficulties in matching to it consequently very great, it does not seem advantageous to discuss it further.

At constant antenna current the fields generated by the coaxial antenna are shown by Moore to become significantly greater as the diameter of the antenna is increased beyond about  $\lambda_1/\pi$ , where  $\lambda_1$  is wavelength in sea water. The effect can be described in terms of an "equivalent current" which increases with diameter of the antenna. For lesser diameters there is no appreciable variation with diameter. In the ELF range, it will be assumed that cross-sectional dimensions equal to or greater than  $\lambda_1/\pi$  are excluded for reasons of physical bulk. The equivalent current used in computing fields for the antenna is then just the same as the actual current in the conductor.

## 6.2 Power Considerations.

Although the gain of the short circuited coaxial antenna can be modified by altering its length in various ways, the effect is one which involves, at most, a few decibels of power in one direction or another from the antenna, and does not

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merit discussion for our purposes. It will be sufficient if we assume for any of the frequencies in the given range that the antenna is a half wavelength or less, so that the gain in any direction differs from that of an isotropic radiator by no more than a factor of two. We will also ignore the directivity gains which can be achieved by the use of multiple radiating elements, or by suppression of alternate phases on a line which is several half wavelengths. It is highly doubtful whether such schemes can effect changes of greater than an order of magnitude or so in power transmitted between two points, without going beyond the limits of physical feasibility.

In the following, then, the equivalent moment of the short circuited coaxial antenna will be taken as  $I_0 l$ , where  $I_0$  is the r.m.s. input current and  $l$  the length of the insulated section. (Radiation from the uninsulated section will be disregarded.)

The impedance of such an antenna is shown by Moore to be defined by:

$$R = \frac{\omega \mu l}{8} , \quad (6.2-1)$$

$$\begin{aligned} X &= \frac{\omega \mu l}{2\pi} \left[ 0.116 - \log \delta b \sqrt{2} + \log b/a \right] \\ &= \frac{\omega \mu l}{2} \left[ 0.116 - \log \delta a \sqrt{2} \right] . \end{aligned} \quad (6.2-2)$$

in which  $\delta = \sqrt{\omega \mu \sigma / 2}$ ,  $b$  is the outer radius of the insulating material of the antenna, and  $a$  the radius of the conductor or antenna core. The tacit assumption has been made that the antenna is sufficiently removed from the boundary between sea

and air so that the impedance is practically the same as that of the antenna in homogeneous sea water.

We will now consider the transmission of power between two short circuited coaxial antennas. The power transmitted is given by

$$P_t = I_t^2 R_t , \quad (6.2-3)$$

where  $I_t$  is the r.m.s. current and  $R_t$  the radiation resistance.

The received power in a receiver matched to the receiving antenna is

$$P_r = \frac{V_r^2}{4R_r} = \frac{(E \ell_r)^2}{4R_r} , \quad (6.2-4)$$

where  $V_r$  is the induced voltage at the terminals of the receiver, and is given by  $E\ell_r$ , with  $E$  the electric field parallel to the axis of the antenna and  $\ell_r$  the receiving antenna length. The receiving antenna radiation resistance is  $R_r$ .

The ratio of received to transmitted power is then given by

$$\frac{P_r}{P_t} = \frac{E^2 \ell_r^2}{4R_r R_t I_t^2} . \quad (6.2-5)$$

Since the field expressions as previously derived are expressed in terms of  $M$ , where

$$M \approx \frac{I_t \ell_t}{4\pi\sigma} , \quad (6.2-6)$$

it will be convenient to write

$$\frac{P_r}{P_t} = \frac{(E^2/M^2) \ell_r^2 \ell_t^2}{64\pi^2 \sigma^2 R_r R_t} \quad (6.2-7)$$

Using the expression for radiation resistance of the short circuited coaxial antenna, this is equivalent to

$$\frac{P_r}{P_t} = \frac{(E^2/M^2) l_r^2 l_t^2}{4\pi \mu^2 \sigma^2 f^2} \quad (6.2-8)$$

Letting  $\sigma = 4$ ,

$$\frac{P_r}{P_t} = 1.01 \times 10^8 \frac{E^2 l_r l_t}{M^2 f^2} \quad (6.2-9)$$

In considering the fields in water due to the mode propagation for either the vertical or horizontal dipole, it may be noted that the radial component is the largest. Because the effect of depth can be computed separately from that of radial distance, the receiving antenna can be assumed to be located at  $z = 0$ , regardless of the fact that the formula for radiation resistance is not then applicable. The attenuation resulting from location at  $z = 0$  can then simply be added to that due to distance.

With alignment of the receiving antenna in the radial direction, we find at 1 cps and 1000 km,

$$\left( \frac{P_r}{P_t} \right)_{1 \text{ cps}}^{(v,m)} = 3.1 \times 10^{-40} l_r l_t e^{-2\alpha z_0} \quad (6.2-10)$$

where  $\alpha$  = imaginary part of  $k_1$ .

The height of the vertical transmitting antenna may well be an important matter, however. This is taken care of by integrating over the range of  $z_0$ .

At 1 cps and 1000 km we have

$$|E_{10}| = 1.75 \times 10^{-24} M \left| \frac{e^{-\alpha z_2} - e^{-\alpha z_1}}{-ik_1} \right| \quad (6.2-11)$$

If the antenna is sufficiently long (say, a half wavelength) so that

$e^{-\alpha z_2} \gg e^{-\alpha z_1}$ , we can write

$$|E_{10}| \approx 1.75 \times 10^{-24} M \left| \frac{e^{-\alpha z_2}}{k_1} \right|. \quad (6.2-12)$$

The power expression then becomes

$$\begin{aligned} \left( \frac{P_r}{P_t} \right)_{1 \text{ cps}}^{(v,m)} &= \frac{3.1 \times 10^{-40} \ell_r}{k_1^2} e^{-2\alpha z_2}, \\ &= 9.78 \times 10^{-36} \ell_r e^{-2\alpha z_2}. \end{aligned} \quad (6.2-13)$$

Letting  $\ell_r$  be a half wavelength (540 meters), we obtain

$$\left( \frac{P_r}{P_t} \right)_{1 \text{ cps}}^{(v,m)} = 5.27 \times 10^{-33} e^{-2\alpha z_2}, \quad (6.2-14)$$

representing a system attenuation of about 323 db, not including losses due to submergence; that is, losses which are a function of  $z$  and  $z_2$  alone. The latter can be calculated readily, and will be ignored for the present.

Let us compare these figures for 1 cps with those which would obtain at 1000 cps. Again with the antenna assumed long,

$$\left( \frac{P_r}{P_t} \right)_{1000 \text{ cps}}^{(v,m)} = 2.68 \times 10^{-34} \ell_r e^{-2\alpha z_2}. \quad (6.2-15)$$

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With a receiving antenna 17 meters (a half wavelength) long, the ratio is

$$4.55 \times 10^{-33} e^{-2\alpha z_2} ,$$

giving a system attenuation of about 323 db excluding water losses.

In the case of the mode solutions for the horizontal dipole we find, under the same assumptions:

$$\left( \frac{P_r}{P_t} \right)_{1 \text{ cps}}^{(h,m)} = 1.22 \times 10^{-26} l_r l_t e^{-2\alpha z_0} , \quad (6.2-16)$$

which for half wavelength antennas and  $\theta = 0$  becomes

$$3.56 \times 10^{-21} e^{-2\alpha z_0} .$$

System attenuation excluding water losses is thus about 204.5 db.

At 1000 cps,

$$\left( \frac{P_r}{P_t} \right)_{1000 \text{ cps}}^{(h,m)} = 7.36 \times 10^{-28} l_r l_t e^{-2\alpha z_0} \cos^2 \theta , \quad (6.2-17)$$

which for a half wavelength and  $\theta = 0$  is

$$2.12 \times 10^{-25} e^{-2\alpha z_0} ,$$

or about 246.7 db attenuation excluding water losses.

It will be of interest to compare these results with the branch line solutions. In all cases the branch line solutions at 1000 km have their greatest magnitudes at 1 cps, and become negligibly small at 1000 cps in comparison with the mode solutions.

The electric field magnitudes are again greatest in the radial direction, although for the horizontal dipole the  $\theta$  component is nearly equal to the radial. In any case, it will suffice to consider only the radial component. Taking first the vertical transmitting antenna, but again with receiving antenna at  $z = 0$  and aligned along a radius, we have

$$\left( \frac{P_r}{P_t} \right)_{1 \text{ cps}}^{(v,b)} = 2.59 \times 10^{-57} \ell_r \ell_t e^{-2\alpha z_0} . \quad (6.2-18)$$

This is negligible compared with the mode solution, and obviously the higher frequencies are of even less interest.

For the horizontal transmitting antenna at 1 cps, under the previous assumptions

$$\left( \frac{P_r}{P_t} \right)_{1 \text{ cps}}^{(h,b)} = 2.22 \times 10^{-31} \ell_r \ell_t e^{-2\alpha z_0} \cos^2 \theta . \quad (6.2-19)$$

With  $\ell_r = \ell_t = 540$  meters and  $\theta = 0$  this becomes

$$6.48 \times 10^{-26} e^{-2\alpha z_0}$$

or about 252 db attenuation not including water losses. The horizontal mode solution gives the least attenuation (205 db) of any considered, but it should be kept in mind that more realistic antenna lengths would cause this to be increased somewhat. With antennas 54 meters long, for example, the system attenuation in this case would be 225 db.

For these figures to have practical significance, they must be related to expected noise power. Unfortunately, the

exact magnitude of atmospheric noise is not well known in the frequency range under discussion. In measurements that have been made<sup>18, 19, 20</sup> noise fields have been given ranging from about 1 to 1800 microvolts per meter per cps for frequencies from 1 to 800 cps. Aarons presents recordings which show noise of the order 0.1 to 250 microvolts per meter per cps, with considerable fluctuations about these values. Willis measured magnetic fluctuations of  $6 \times 10^{-8}$  gauss per cps at 5 cps, ranging down to  $4 \times 10^{-11}$  gauss per cps at 800 cps. This gives from 1800 to 1.2 microvolts per meter vertical electric field. Goldberg mentions magnetic components of the order  $3 \times 10^{-7}$  gauss, which would be 9000 microvolts per meter electric field, but it appears--without being entirely clear--that this is for his entire pass band of 1.0 to 150 cps. If uniform across the band this would represent noise strength of 60 microvolts per meter per cps.

Since the most detailed--although somewhat difficult to interpret--results are those presented by Aarons, we will take the figure 2 microvolts/meter/cps from one of his charts as being roughly representative at frequencies near the low end of the 1 - 1000 cps range. It should be kept in mind that this may be extremely optimistic. We will consider a transmitter of 100 kilowatts output power and a frequency of 1 cps,

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<sup>18</sup>J. Aarons, Journal of Geophysical Research, 61, 647 (1956)

<sup>19</sup>H. F. Willis, Nature, 161, 887 (1948)

<sup>20</sup>P. A. Goldberg, Nature, 177, 1219 (1956)

with radial distance 1000 km between transmitter and receiver. Assuming  $\theta = 0$  and half wave (540 meter) antennas, the received power at  $z = 0$  will be, essentially,

$$P_r^{(h,m)} = 3.56 \times 10^{-16} e^{-2\alpha z_0} . \quad (6.2-20)$$

When the vertical electric field in air is 2 microvolts per meter, the magnitude of the tangential magnetic field at the water surface will be

$$|H_{\tan}| = \frac{|E_{\text{vert}}|}{\sqrt{\mu_0/\epsilon_0}} = \frac{|E_{\text{vert}}|}{377} \quad (6.2-21)$$

The tangential electric field will be given by

$$|E_{\tan}| = |Z_0| |H_{\tan}| ,$$

where  $Z_0 = (1+i) \frac{\sqrt{\omega\mu}}{2\sigma}$  . Therefore,

$$\begin{aligned} |E_{\tan}| &= \frac{1.4 \times 10^{-3} \sqrt{f}}{377} |E_v| , \\ &= 3.76 \times 10^{-6} \sqrt{f} |E_v| . \end{aligned} \quad (6.2-22)$$

The figure 2  $\mu\text{v}/\text{meter}$  at 1 cps then gives

$$|E_{\tan}| = 7.5 \times 10^{-2} \text{ volts/meter/cps} .$$

Received noise power with the antenna aligned for maximum

$|E_{\tan}|$  is:

$$P_r = \frac{|E_{\tan}|^2 \ell_r^2}{4R_r} = \frac{2.5 \times 10^5 E_r^2 \ell_r^2}{f} \quad (6.2-23)$$

which for the above figures becomes

$$P_r = 7.63 \times 10^{-15} \text{ watts} .$$

Thus the signal to noise ratio will be

$$\left( \frac{S}{N} \right)_{\substack{(h,m) \\ 1 \text{ cps} \\ 1000 \text{ km}}} = \frac{3.56 \times 10^{-16}}{7.63 \times 10^{-15}} e^{-2\alpha z_0} = 4.66 \times 10^{-2} e^{-2\alpha z_0} , \quad (6.2-24)$$

or about - 13.3 db. The number of decibels of attenuation due to the depth of the transmitting antenna will of course subtract directly from this, that is, will be deleterious to the S/N ratio. As mentioned previously, the figure taken for noise magnitude is probably optimistic to begin with.

It may be noted here that receiver thermal noise is not important compared with atmospheric noise in this case. For example if one considers the receiver as matched to the antenna, then for 1 cps bandwidth the noise power in the receiver input resistance is  $2kT$  ( $k$  = Boltzmann's constant,  $T$  = temperature, degrees Kelvin). For  $20^{\circ}$  C this is  $8.03 \times 10^{-21}$  watts, which is about 250 decibels below the transmitted power of  $10^5$  watts. Thus the basic limitation is evidently that associated with natural noise.

### 6.3 Antenna Depth Considerations.

For the mode solutions, the fields increase monotonically with frequency, but with a dependence on frequency that is somewhat involved. Moreover, the dependence is different for different components of the field, and varies with distance.

In Figures 6.1 and 6.2, the field components are drawn as a function of frequency on logarithmic graphs. As a very rough approximation at 1000 km distance, the dependence might be taken to be (aside from the exponential term) for a radial component:

$$\begin{aligned} \text{Low Frequency: } E_{10}^{(v,m)} &\sim f^{3/2}; \quad E_{10}^{(h,m)} \sim f^{0.2} \\ \text{High Frequency: } E_{10}^{(v,m)} &\sim f^2; \quad E_{10}^{(h,m)} \sim f^{3/2} \end{aligned} \quad (6.3-1)$$

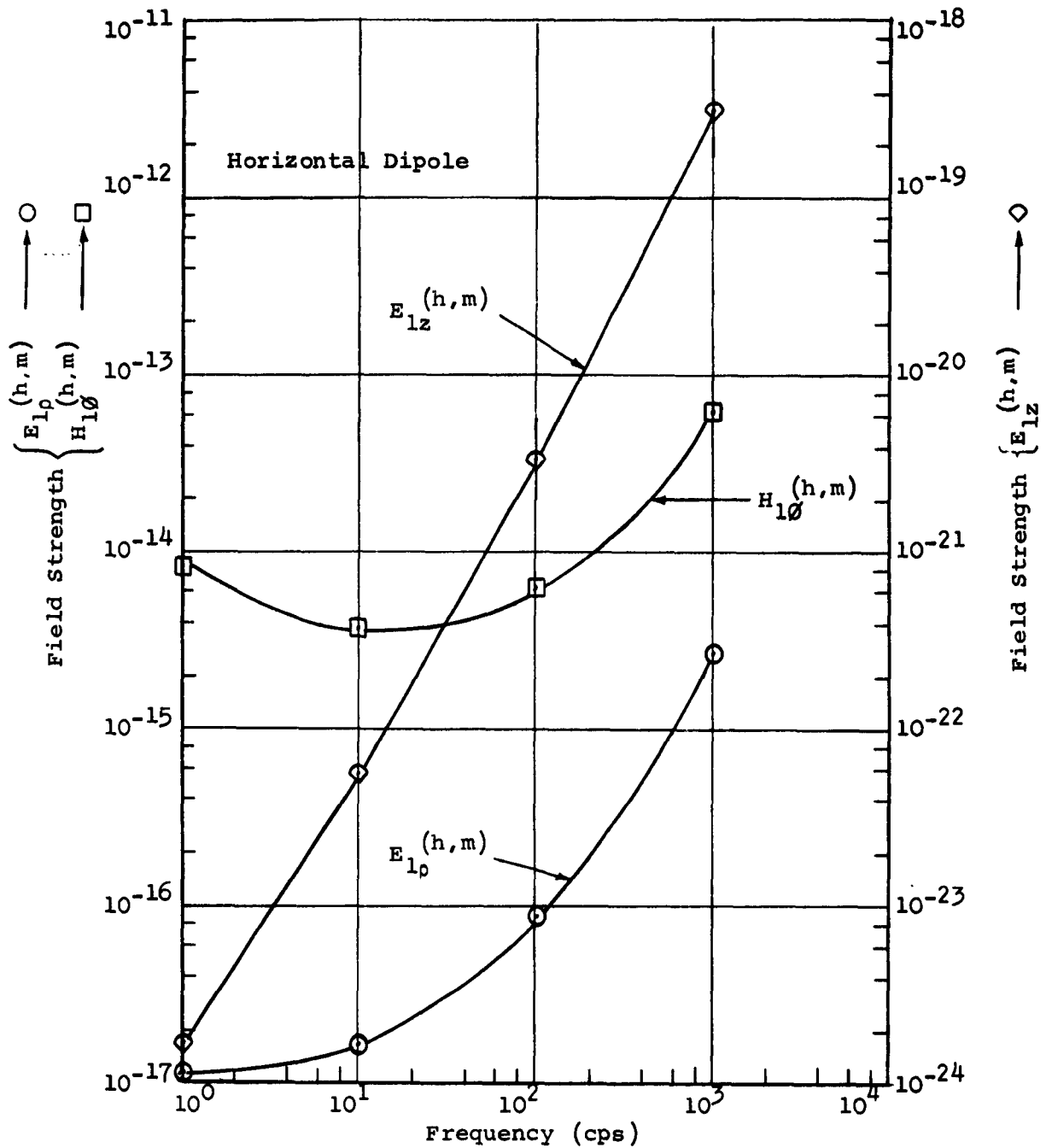


Figure 6.1

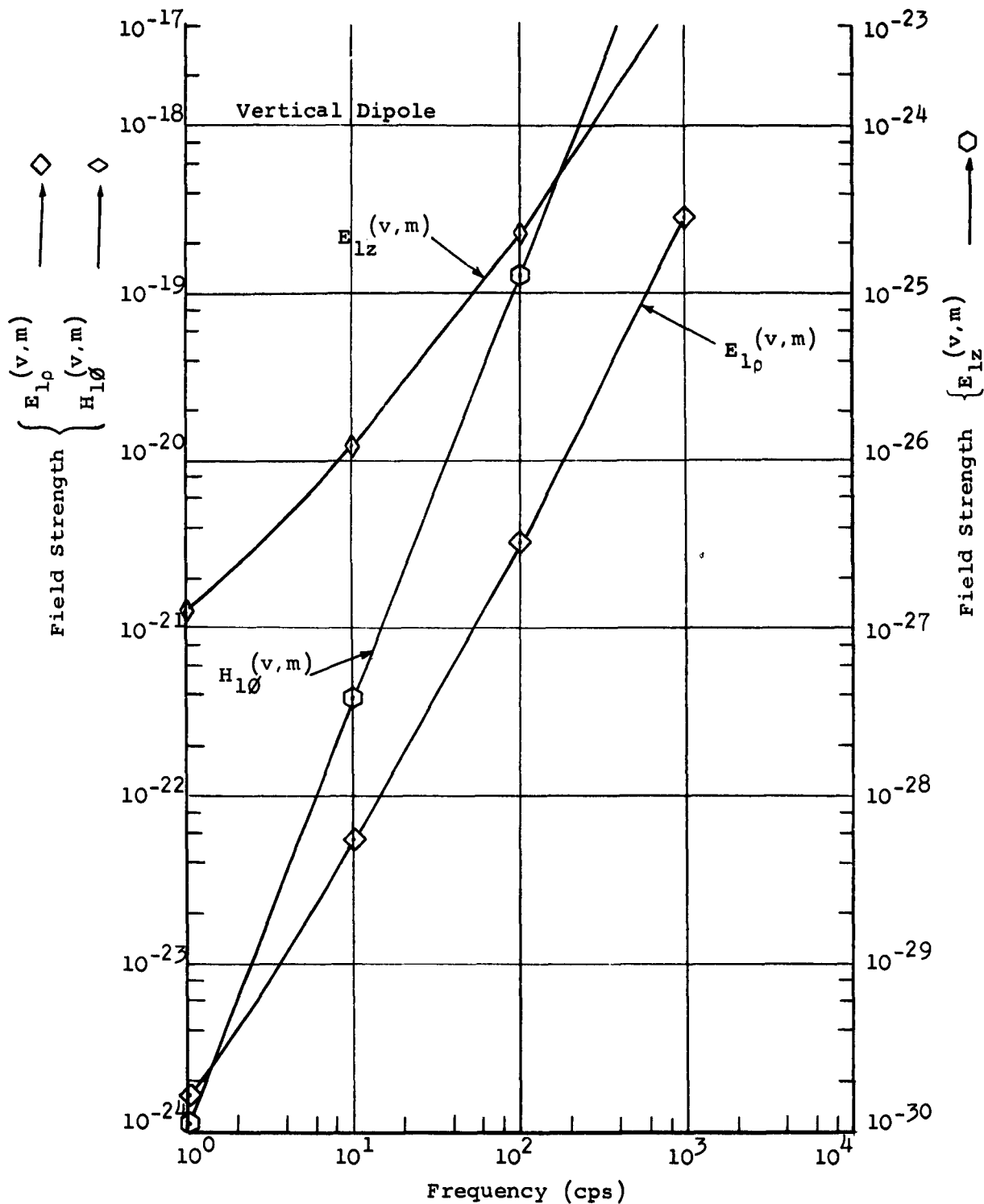


Figure 6.2

Let  $p$  be the power of frequency involved as a very rough approximation. The optimum frequency for a given field component will be given by differentiating the field component with respect to  $f$  and setting the result equal to zero. We thus obtain (assuming receiving antenna depth is zero)

$$pf^{p-1} e^{-\alpha z_0} - iz_0 f^p \frac{dk_1}{df} = 0 \quad (6.3-2)$$

From this it is found that

$$f_{opt} = \frac{1.22 p \times 10^5}{z_0^2} \quad (6.3-3)$$

For the  $(h,m)$  solution and a receiving antenna at  $z < 0$ ,  $z_0^2$  in this equation can be replaced simply by  $(z-z_0)^2$ , so that the optimum frequency, to a crude approximation, varies inversely as the square of the combined depth of receiving and transmitting antennas.

The same relationship was found by Moore to hold in a more exact fashion when the atmosphere is considered homogeneous.

The  $z$ -dependence for the  $(v,m)$  situation is more complicated, but essentially of exponential behavior, so that again an approximate idea of optimum frequency can be obtained in the same manner.

As an example, if the combined depth of receiving and transmitting antennas is 100 meters, one finds the optimum frequency is about  $12.2 p$  for a distance of 1000 km. Considering the  $E_{10}$  component, this would mean  $f_{opt}$  would be roughly 20 cps for the vertical transmitting antenna. For the horizontal case,  $f_{opt}$  would be somewhere in the vicinity of 2.5 to 5.0 cps; and not very sharply defined on account of the near-zero frequency

dependence. Evidently quite low frequencies must be used when the antenna depths are appreciable.

The branch line solutions, on the other hand, decrease monotonically with frequency. For them there is no optimum, and the "best" frequency is the lowest that can be used.

#### 6.4 Summary on Antennas.

In the frequency range 1 - 1000 cps, the system attenuation for a distance of 1000 kilometers between submerged horizontal short-circuited coaxial antennas is large but does not preclude the possibility of practical communication. The noise situation is not well known, but at best it would appear that signal will be about 13 db or more below noise. Further degradation of the signal-noise ratio will occur due to vertical travel of the waves between antennas and water surface. The quoted 13 db figure is that obtained for the mode solution for the horizontal transmitting antenna at a frequency of 1 cps.

At distances less than 1000 km the so-called "branch line" solution becomes comparable in magnitude to the mode solution, but unfortunately our analysis does not reveal what sort of behavior can be expected close to the antenna for the "branch line" solutions. One method of tackling this question would be by means of numerical integration, possibly programmed for an automatic computer. A possible approach would be to draw a branch cut from  $k_3$  toward  $k_1$  so that Bessel functions of argument  $(e^{-i\pi/4}x)$  could be used. Close to  $k_1$ , the contribution to the integral will be negligible for all except such low

values of  $\rho\sqrt{f}$  as to be essentially without interest. This is because the Hankel function in the vicinity of  $k_1$  becomes proportional to  $e^{-ik_1\rho}$ , provided (roughly)  $\rho\sqrt{f} \gtrsim 200$ . Whether this approach is better than that of putting the branch cut vertically downward would have to be determined by a more detailed study.

The branch line solution is much smaller than the mode solution at 1000 km distance and 1 cps and becomes less important for higher frequencies at this same distance.

The very slight attenuation with distance of the mode solution would appear to make it ideal for communications purposes. However, it requires a very efficient submerged antenna to launch a wave up through the sea-air interface before it even goes between the sea and the ionosphere. It has been shown for a practical length antenna of 54 meters that possible communications would be at a range much less than 1000 km.

One does not exclude a slight possibility that a new and perhaps radical antenna design might enable much more efficient coupling of electromagnetic energy to the water. This possibility has been dubbed "slight", however, because the physical nature of the problem seems to mitigate against it in every way. The requirement of physical transportability is in direct conflict with the requirement that low current densities be maintained in the vicinity of the antenna. A very analogous situation is one which arises in acoustics, namely that of matching the high impedance of an amplifier-loudspeaker combination to the low impedance of air. The problem is particularly acute at long

wavelengths and is met by using one or more loudspeakers which are as large in area as is physically and economically feasible. Even then, it is customary to enhance the impedance match by using baffles or horns of various types--thereby increasing even more the effective area of energy coupling. It is the opinion of the writer that the requirement of large size is just as basic to the submerged antenna problem as it is to this.

The same remarks are applicable to the receiving antenna. Because of the impedances involved, the efficiency with which it intercepts electrical energy in the water must depend on its physical size.

## APPENDIX A

When the distance between perfectly conducting planes of a parallel plane waveguide is less than a half wavelength, the propagation constants for TE and TM modes are pure imaginary. In the development of this paper, their location would be represented by poles on the negative imaginary axis of the complex  $\lambda$ -plane.

To obtain an estimate as to the amount by which such poles may move when finite conductivity is introduced in the boundaries, we will consider a rather idealized case where a wave of infinite transverse extent is assumed to be propagating between the infinite parallel planes. The conductivity of both planes will be taken as equal to that of the ionosphere; thus it is to be expected that our results will represent a worse situation than the one we have been investigating.

The direction of propagation will be taken along the  $z$ -axis. The planes are the surfaces  $y = 0$  and  $y = h$ . For TM waves we can assume a magnetic field of the type

$$H_x = H_0 \cos k_y y e^{i\omega t - \gamma z} \quad (A-1a)$$

and derive the remaining field components by use of Maxwell's equations. It is found that:

$$E_z = \frac{k_y}{i\omega\epsilon} H_0 \sin(k_y y) e^{i\omega t - \gamma z} \quad (A-1b)$$

$$E_y = -\frac{\gamma}{i\omega\epsilon} H_0 \cos(k_y y) e^{i\omega t - \gamma z} \quad (A-1c)$$

All other field components are zero. From the wave equation one obtains the relation

$$k_y^2 = \gamma^2 + \omega^2/c^2 \quad (A-2)$$

If the walls were perfectly conducting, it would be necessary that  $k_y$  have the eigenvalues  $n\pi/h$ ,  $n = 1, 2, 3, \dots$ , in order for the boundary conditions to be satisfied. In this case we will assume that the conductivity is sufficiently good so that the eigenvalues are perturbed only by a small amount  $p$ . Then

$$k_y \approx \frac{n\pi}{h} + p \quad (A-3)$$

and to terms of order  $p$ ,

$$k_y^2 \approx \left(\frac{n\pi}{h}\right)^2 + \frac{2n\pi p}{h} \quad (A-4)$$

For the fields in the conducting medium we can take

$$\begin{aligned} H_x &= C e^{-by-\gamma z+i\omega t} \\ E_z &+ b/\sigma C e^{-by-\gamma z+i\omega t} \\ E_y &= -\gamma/\sigma C e^{-by-\gamma z+i\omega t} \end{aligned} \quad (A-5)$$

where satisfaction of the wave equation requires that

$$b^2 + \gamma^2 - i\omega\mu\sigma = 0 \quad (A-6)$$

Here the displacement current has been assumed negligible in comparison with the conduction current. This is valid for  $\sigma/\omega\epsilon \gg 1$ , which is true in our case.

At  $y = b$  the boundary condition requires

$$\frac{k_y}{i\omega\epsilon} H_0 \sin(k_y h) e^{i\omega t - \gamma z} = \frac{b}{\sigma} C e^{-bh - \gamma z + i\omega t} \quad (A-7a)$$

$$H_0 \cos(k_y h) e^{i\omega t - \gamma z} = C e^{-bh - \gamma z + i\omega t} \quad (A-7b)$$

Dividing the top equation by the bottom one:

$$\frac{k_y \sin(k_y h)}{\cos(k_y h)} = \frac{i\omega\epsilon b}{\sigma} \quad (\text{A-8})$$

The mode least attenuated will be that for which  $n = 1$ ;  
hence

$$k_y = \frac{\pi}{h} + p$$

$$\left(\frac{\pi}{h} + p\right) \tan \left[\left(\frac{\pi}{h} + p\right) h\right] = \frac{i\omega\epsilon b}{\sigma} \quad (\text{A-9})$$

Approximately,

$$\left(\frac{\pi}{h} + p\right) [\tan \pi + ph \sec^2 \pi] = \frac{i\omega\epsilon b}{\sigma} \quad (\text{A-10})$$

from which, to terms of order  $p$ ,

$$p \approx \frac{i\omega\epsilon b}{\pi\sigma} \quad (\text{A-11})$$

We can write

$$b = \sqrt{i\omega\mu\sigma - \left(\frac{\pi}{h}\right)^2 - \frac{2p\pi}{h} + \frac{\omega^2}{c^2}} \quad (\text{A-12})$$

For frequencies well into the "stop band", that is, well below the so-called cutoff frequency, the last two terms can be neglected, so

$$b \approx \sqrt{i\omega\mu\sigma - \left(\frac{\pi}{h}\right)^2} \quad (\text{A-13})$$

Therefore,

$$\gamma \approx \sqrt{\left(\frac{\pi}{h}\right)^2 - \frac{\omega^2}{c^2}} + i \frac{2\omega\epsilon}{h\sigma} \sqrt{i\omega\mu\sigma - \left(\frac{\pi}{h}\right)^2} \quad (\text{A-14})$$

If the conductivity is allowed to become infinite, obviously one recovers the expression

$$\gamma = \sqrt{\left(\frac{\pi}{h}\right)^2 - \frac{\omega^2}{c^2}} \quad (\text{A-15})$$

which is the usual equation in such a case. We shall now inquire into the magnitude of what we might call the "correction term",  $C_0$ :

$$C_0 = i \frac{2\omega\epsilon}{h\sigma} \sqrt{i\omega\mu\sigma - \left(\frac{\pi}{h}\right)^2} \quad (A-16)$$

If this is small in comparison with  $\left[\left(\frac{\pi}{h}\right)^2 - \frac{\omega^2}{c^2}\right]$ , we can be satisfied that a pole in the complex plane representing the location of  $\gamma$  will not be greatly shifted from its value for perfect conductivity.

It will suffice to check the frequencies  $f = 1$  and 1000 cps. For  $f = 1$ ,

$$C_0 \approx - \frac{2\omega\epsilon\pi}{\sigma h^2} \quad (A-17)$$

and

$$\gamma \approx \frac{\pi}{h} \left(1 - \frac{\omega\epsilon}{\sigma h}\right) \approx \frac{\pi}{h} (1 - 6.17 \times 10^{-11}) \quad (A-18)$$

Evidently  $\gamma$  is not changed very much in this case. For  $f = 1000$ :

$$C_0 \approx - 3.52 \times 10^{-7} (1-i), \quad (A-19)$$

and one obtains

$$\gamma \approx 2.85 \times 10^{-5} - (1-i) 3.52 \times 10^{-7} \quad (A-20)$$

so that again the perturbation is seen to be slight.

For TE waves we can carry out a similar development.

The fields in air can be written

$$\begin{aligned} E_x &= E_0 \sin(k_y y) e^{-\gamma z + i\omega t} \\ H_y &= \frac{\gamma}{i\omega\mu} E_0 \sin(k_y y) e^{-\gamma z + i\omega t} \\ H_z &= \frac{k_y}{i\omega\mu} E_0 \cos(k_y y) e^{-\gamma z + i\omega t} \end{aligned} \quad (A-21)$$

-v-

and those in the conductor are:

$$\begin{aligned} E_x &= A e^{-by-\gamma z+i\omega t} \\ H_y &= A \frac{\gamma}{i\omega\mu} e^{-by-\gamma z+i\omega t} \\ H_z &= -A \frac{b}{i\omega\mu} e^{-by-\gamma z+i\omega t} \end{aligned} \quad (A-22)$$

At the boundary  $y = h$ ,

$$E_o \sin(k_y h) = A e^{-bh} \quad (A-23)$$

$$\frac{k_y}{i\omega\mu} E_o \cos(k_y h) = -A \frac{b}{i\omega\mu} e^{-bh} \quad (A-24)$$

Dividing the top equation by the bottom one:

$$\tan(k_y h) = -\frac{k_y}{b} \quad (A-25)$$

Again letting  $k_y = \frac{\pi}{h} + p$  for the  $n = 1$  mode,

$$\begin{aligned} \tan h \left( \frac{\pi}{h} + p \right) &= \tan ph = -\frac{1}{b} \left( \frac{\pi}{h} + p \right) \\ b &= \sqrt{i\omega\mu\sigma - \left( \frac{\pi}{h} + p \right)^2} \quad \left( \text{dropping } \frac{\omega^2}{c^2} \text{ term} \right) \\ &= i \left( \frac{\pi}{h} + p \right) \sqrt{1 - \frac{i\omega\mu\sigma}{2 \left( \frac{\pi}{h} + p \right)^2}} \\ b &\approx i \left( \frac{\pi}{h} + p \right) \left[ 1 - \frac{i\omega\mu\sigma}{2 \left( \frac{\pi}{h} + p \right)^2} \right] \end{aligned} \quad (A-26)$$

Then

$$\tan ph \approx i \left[ 1 + \frac{i\omega\mu\sigma h^2}{2(\pi+ph)^2} \right] \quad (A-27)$$

Let  $ph = ich$ , then

$$\begin{aligned} \tan ph &= i \tanh ch \\ \tanh ch &= 1 + \frac{i\omega\mu\sigma h^2}{2(\pi+ich)^2} \end{aligned} \quad (A-28)$$

Using the relation

$$\tanh ch = \frac{1 - e^{-2ch}}{1 + e^{-2ch}} \quad (\text{A-29})$$

we obtain

$$e^{-2ch} = \frac{i\omega\mu\sigma h^2}{4(\pi + ich)^2 + i\omega\mu\sigma h^2} \quad (\text{A-30})$$

To solve, it will be necessary to apply an iteration procedure, using numerical values. At  $f = 1$  cps,  $i\omega\mu\sigma h^2 = i(0.64)$ .

Guessing the denominator to have the value 40, we have

$$\begin{aligned} e^{-2ch} &= -\frac{i0.64}{40} \\ 2ch &= \log 40 - \log 0.64 + i\pi/2 \\ c &= 1/h (2.285 + i0.785) = 1/h(2.41 \exp i18.9^\circ) . \end{aligned} \quad (\text{A-31a})$$

Using this value for  $c$  in the right hand side of (A-30), we obtain as the next approximation to  $c$ :

$$c = 1/h (2.1 + i1.56) = 1/h (2.61 \exp i36.6^\circ) \quad (\text{A-31b})$$

The next iteration gives

$$c = 1/h (1.88 + i1.71) = 1/h (2.54 \exp i42.3^\circ), \quad (\text{A-31c})$$

the next

$$c = 1/h (1.774 + i1.705) = 1/h (2.46 \exp i43.8^\circ), \quad (\text{A-31d})$$

and the next

$$c = 1/h (1.739 + i1.675) = 1/h (2.41 \exp i43.9^\circ). \quad (\text{A-31e})$$

Evidently we are coming close to a good approximation, and it will suffice to use this last figure. Since  $p = ic$ ,

$$\begin{aligned} p &= i/h (2.41 \exp(i43.9^\circ)) = 1/h (2.41 \exp(i133.9^\circ)) \\ &= \pi/h (.767 \exp(i133.9^\circ)) \end{aligned}$$

and

$$\begin{aligned} \gamma &= \pi/h + p \\ \gamma &= \pi/h [1 + .767 \exp(i133.9^\circ)] \\ \gamma &= \pi/h [.468 + i.552] \\ \gamma &= \pi/h [.723 \exp(i49.7^\circ)] . \end{aligned}$$

Thus the magnitude of  $\gamma$ , which at 1 cps would be nearly  $\pi/h$  for perfectly conducting boundaries, is not greatly modified in this case. The actual attenuation is reduced from  $\pi/h$  to  $0.468 \pi/h$ , and a phase shift of  $0.552 \pi/h$  radians per meter is introduced. In the  $\lambda$ -plane, this corresponds to the pole moving from  $\lambda_1 = -\frac{i\pi}{h}$  to  $\lambda_2 = \frac{\pi}{h} (0.552) - \frac{i\pi}{h} (0.468)$ .

At  $f = 1000$  cps,  $i\omega\mu\sigma \gg (\frac{\pi}{h} + p)^2$ . It is easily found that

$$\begin{aligned} p &\approx -\frac{\pi}{h} \frac{1}{1 + \sqrt{1} 25.3} \\ &= (1-i) 9.8 \times 10^{-7} \end{aligned}$$

which compared with  $2.85 \times 10^{-5}$  is fairly small.

We can summarize the above by saying that if TE and TM modes were excited in the case of perfectly conducting boundaries (for the frequency range 1-1000 cps and  $h = 9 \times 10^4$  meters), their amplitudes would be attenuated with distance much more severely than the TEM mode. Hence one would expect that they would not be significant except within short distances from the antenna. It has been shown that this conclusion is still applicable when the boundaries are assigned a conductivity of  $10^{-5}$  mhos/meter.

## APPENDIX B

The equations involved in the calculations of the field components are presented in a simplified form to facilitate their calculation. In these formulae, terms which are negligible compared with others, over the frequency range 1-1000 cps have been discarded. The range of validity of these formulae is further specified by parameter values listed under (2.5-20). For example, one cannot let  $h$  tend to infinity without invalidating some conditions under which the formulae are obtained.

In this way, it has been found that

$$\Gamma(\lambda_o) \approx \frac{i\delta_1 k_3 h}{k_1(1+ik_3 h)} \quad (B-1)$$

or in terms of more fundamental parameters

$$\Gamma(\lambda_o) \approx \frac{1 \sqrt{\frac{\sigma_3}{\sigma_1}} h}{\omega \epsilon_o \left[ 1 + h(1+i) \sqrt{\frac{\omega \mu_o \sigma_3}{2}} \right]} \quad (B-2)$$

Another quantity involved in the calculations is approximated by

$$(k_1 + z_2)'_{\lambda=\lambda_o} \approx - \frac{i2\lambda_o \delta_1 \delta_3 h}{\sigma_1 (\delta_3 - \beta_3 h)} \quad (B-3)$$

In this formula, the second term in the parenthesis, in the denominator, is negligible at lower frequencies and affects only the second significant figure at 1000 cps. If it is dropped, the formula reduces to

$$(k_1 + z_2)'_{\lambda=\lambda_o} \approx - \frac{i2\lambda_o h}{\omega \epsilon_o} \quad (B-4)$$

Therefore the mode solution field components in region 1 arising from the vertical electric dipole can be written approximately as

$$E_{1\rho}^{(v,m)} \approx Idl \frac{i\lambda_o k_1 \omega \epsilon_o}{4\sigma_1^2 h} H_1^{(2)}(\lambda_o \rho) e^{ik_1(z-z_o)} \quad (B-5)$$

$$E_{1z}^{(v,m)} \approx Idl \frac{\lambda_o^2 \omega \epsilon_o}{4\sigma_1^2 h} H_o^{(2)}(\lambda_o \rho) e^{ik_1(z-z_o)} \quad (B-6)$$

$$H_{1\phi}^{(v,m)} \approx Idl \frac{\lambda_o \omega \epsilon_o}{4\sigma_1 h} H_1^{(2)}(\lambda_o \rho) e^{ik_1(z-z_o)} \quad (B-7)$$

in which M has been replaced by

$$\frac{Idl}{4\pi\sigma_1}$$

and Idl is taken to be the moment of the elemental dipole.

The formulae for mode solutions in region 2 are likewise readily simplified in this way.

In the case of the horizontal electric dipole, the following simplified mode solutions can be written, using the approximations previously indicated:

$$E_{1\rho}^{(h,m)} \approx -Idl \frac{\lambda_o^2 k_3}{8\sigma_1(1 + ik_3 h)} \cos \phi [H_o^{(2)}(\lambda_o \rho) - H_2^{(2)}(\lambda_o \rho)] e^{ik_1(z-z_o)} \quad (B-8)$$

$$E_{1z}^{(h,m)} \approx Idl \frac{i\lambda_o^3 k_3 H_1^{(2)}(\lambda_o \rho)}{4\sigma_1 k_1(1 + ik_3 h)} \cos \phi e^{ik_1(z-z_o)} \quad (B-9)$$

$$H_{1\phi}^{(h,m)} \approx Idl \frac{i\lambda_o^2 k_3 [H_o^{(2)}(\lambda_o \rho) - H_2^{(2)}(\lambda_o \rho)]}{8k_1(1 + ik_3 h)} \cos \phi e^{ik_1(z-z_o)} \quad (B-10)$$

In region 2,  $S(\lambda, z)$ , defined subsequent to equation (4.1-11) can be written

$$S(\lambda, z) = -\lambda \Gamma(\lambda) f_1(\lambda, z) \quad (B-11)$$

where  $f_1(\lambda, z) = \beta_1 \cos \beta_2 z + \delta_1 \beta_2 \sin \beta_2 z$ .

$$\text{Also } r(\lambda, z) = -\lambda \Gamma(\lambda) f_2(\lambda, z) \quad (B-12)$$

where  $f_2(\lambda, z) = \beta_1 \sin \beta_2 z - \delta_1 \beta_2 \cos \beta_2 z$ .

The mode solution field components in region 2 can therefore be approximated by

$$E_{2\rho}^{(h,m)} \approx \text{Idl} \frac{\lambda_o^2 k_3 e^{-ik_1 z_o \cos \varnothing}}{8\sigma_1 k_1 (1 + ik_3 h)} [H_o^{(2)}(\lambda_{o\rho}) - H_2^{(2)}(\lambda_{o\rho})] f_1(\lambda_o, z) \quad (B-13)$$

$$E_{2z}^{(h,m)} \approx -\text{Idl} \frac{\lambda_o^3 k_3 e^{-ik_1 z_o \cos \varnothing}}{4\sigma_1 \beta_2^{(0)} k_1 (1 + ik_3 h)} H_1^{(2)}(\lambda_{o\rho}) f_2(\lambda_o, z) \quad (B-14)$$

$$H_{2\varnothing}^{(h,m)} \approx -\text{Idl} \frac{i\omega \epsilon_o \lambda_o^2 k_3 e^{-ik_1 z_o \cos \varnothing}}{8\sigma_1 \beta_2^{(0)} k_1 (1 + ik_3 h)} [H_o^{(2)}(\lambda_{o\rho}) - H_2^{(2)}(\lambda_{o\rho})] f_2(\lambda_o, z) \quad (B-15)$$

with values of the parameters as given in 2.5-20) it is found that, as a function of frequency

$$\beta_2^{(0)} \approx 2.34 \times 10^{-8} f^{3/4} e^{-i 5\pi/8} \quad (B-16)$$

The above approximate formulae represent a considerable simplification over the complete exact equations and should greatly facilitate any computation which might need to be made concerning the field components.

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